Spectral Graph Theory and Applications **WS 2011/2012**

Problem Set 1

Due: Nov. 25

Note: Please send the solutions to `hsun@mpi-inf.mpg.de` or `sauerwal@mpi-inf.mpg.de` before the deadline.

Problem 1 Let G be any n-vertex graph and define

$$\alpha = \min_{S\colon |S|\leq n/2} \frac{|\Gamma(S) \setminus S|}{|S|}.$$

Prove that for any subset $S \subseteq V$,

$$|\Gamma(S) \setminus S| \geq (\alpha/2) \cdot \min\{|S|, |V \setminus S|\}.$$

Problem 2 Let $G = (V, E)$ be any undirected graph with Laplacian matrix $\mathbf{L}$ and $f : V \to \mathbb{R}$ be any function. Prove that

$$f^{\mathrm{T}}\mathbf{L}f = \sum_{\{u,v\}\in E} (f(u) - f(v))^2 .$$

Problem 3 Let G be any graph and consider its Laplace Matrix $\mathbf{L}$. Prove that the number of connected components of G is equal to the multiplicity of the eigenvalue 0.

Problem 4 For any integer $d \in \mathbb{N}$, consider the d-dimensional hypercube $H = (V, E)$ defined as $V = \{0,1\}^d, n = 2^d$ and $E := \{\{u, v\}\colon u, v \in V \wedge |u - v|_1 = 1\}$. Prove that for any bitstring $x \in \{0,1\}^d$, the vector $v(x)$ defined as

$$(v(x))_u = (-1)^{\langle u,x\rangle}$$

is an eigenvector of $\mathbf{A}$ whose eigenvalue equals the number of ones in x (note that in the above notation, we replace the coordinates $1, 2, \ldots, n$ for u of the vector $v(x)$ by the corresponding bitstrings).