Lecture 3: Expander Mixing Lemma

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1 Existence and Constructibility of Expander Graphs

Expander graphs have two seemingly contradictory properties: low degree and high connectivity. Two general problems are *existence* and *constructibility* of expander graphs. Among these two problems, existential proofs of expanders are easier, as one can resort to probabilistic techniques. Further, the existence of expanders can be often used as a black-box to show the existence of other interesting combinatorial objects. On the other hand, many applications of expanders need explicit constructions. We will mention some explicit constructions in this lecture, but they do not always match the bounds given by probabilistic methods.

Let $\mathcal{G}_{d,N}$ be the set of bipartite graphs with bipartite sets L, R of size N and left degree d. The following lemma shows the existence of expanders.

Theorem 3.1. *For any d, there exists an $\alpha(d) > 0$, such that for all N*

$$\Pr[G \textit{ is an } (\alpha N, d-2)\textit{-expander }] \geq 1/2,$$

where G is chosen uniformly from $\mathcal{G}_{d,N}$.

Proof. Define

$$p_k := \Pr\left[\exists S \subseteq L : |S| = k, |\Gamma(S)| < (d-2)|S|\right].$$

So G is not an $(\alpha N, d-2)$-expander iff $\sum_k p_k > 0$.

Assume that there is a set S of size K and $|\Gamma(S)| < (d-2)|S|$. Then there are at least $2k$ repeats among all the neighbors of vertices in S. We calculate the probability

$$\Pr[\text{at least } 2k \text{ repeats among all the neighbors of vertices in } S] \leq \binom{dk}{2k}\left(\frac{dk}{N}\right)^{2k}.$$

Therefore

$$\begin{aligned} p_k &\leq \binom{N}{k}\binom{dk}{2k}\left(\frac{dk}{N}\right)^{2k} \\ &\leq \left(\frac{N\mathrm{e}}{k}\right)^{k} \cdot \left(\frac{dk\mathrm{e}}{2k}\right)^{2k} \cdot \left(\frac{dk}{N}\right)^{2k} \\ &= \left(\frac{cd^4k}{N}\right)^{k} \end{aligned}$$

where $c = \mathrm{e}^3$. By setting $\alpha = 1/(cd^4)$ and $k \leq \alpha N$, we know that $p_k \leq 4^{-k}$ and

$$\Pr[G \text{ is not an } (\alpha N, d-2)\text{-expander}] \leq \sum_{k=1}^{\alpha N} p_k \leq \sum_{k=1}^{\alpha N} 4^{-k} \leq 1/2.$$

□

Let us now turn to the constructibility of expanders.

Definition 3.2. *Let $\{G_i\}_{i\in\mathbb{N}}$ be a family of expander graphs where G_i is a d-regular graph on n_i vertices and the integers $\{n_i\}$ are increasing, but not too fast.(e.g. $n_{i+1} \le n_i^2$ will do)*

1. *The family is called **Mildly Explicit** if there is an algorithm that generates the j-th graph in the family $\{G_i\}_{i\in\mathbb{N}}$ in time polynomial in j.*
2. *The family is called **Very Explicit** if there is an algorithm that on input of an integer i, a vertex $v \in V(G_i)$ and $k \in \{1,\cdots,d\}$ computes the k-th neighbor of the vertex v in the graph G_i. The algorithm's running time should be polynomial in its input length.*

Example. *A family of 3-regular p vertex graph for a prime number p. Let $G = (\mathbb{Z}_p, E)$. For any vertex $x \in \mathbb{Z}_p$, vertex x is connected to $x+1, x-1$ and x^{-1}. (The inverse of 0 is defined to be 0.)*

The family of expanders above is just mildly explicit, since we are at present unable to generate large prime number deterministically.

Example (Margulis, 1973)**.** *Fix a positive integer M and let $[M] = \{1,2,\cdots,M\}$. Define the bipartite graph $G = (V,E)$ as follows. Let $V = [M]^2 \cup [M]^2$, where vertices in the first partite set as denoted $(x,y)_1$ and vertices in the second partite set are denoted $(x,y)_2$. From each vertex $(x,y)_1$, put in edges*

$$(x,y)_2, (x,x+y)_2, (x,x+y+1)_2, (x+y,y)_2, (x+y+1,y)_2,$$

where all arithmetic is done modulo M. Then G is an expander.

Example (Jimbo and Maruoka, 1987)**.** *Let $G = (L \cup R, E)$ be the graph described above, then $\forall X \subset L$, $|\Gamma(X)| \ge |X|(1 + d_0|\overline{X}|/n)$, where $d_0 = (2-\sqrt{3})/4$ is the optimal constant.*

2 Expander Mixing Lemma

Consider two experiments on a d-regular graph G. (1) Pick a random vertex $u \in V$ and then pick one of its neighbors v. (2) Pick two random vertices $u, v \in V$ randomly and independently from $V \times V$. What is the probability of the event $(u,v) \in S \times T$, $S,T \subseteq V$ for these two experiments? For the first experiment, the probability is $|E(S,T)|/(nd)$. The probability for the second probability is $\mu(S)\cdot\mu(T)$, where $\mu(S) := |S|/n$ is the density of set S.

For the random bits used in these two experiments, it is easy to show that the first experiment uses $\log n + \log d$ random bits and the second one uses $2\log n$ random bits. However, we will show that for graphs with good expansion these two probabilities are quite close to each other.

Lemma 3.3 (Expander Mixing Lemma)**.** *[AC88] Let $G = (V,E)$ be a d-regular n-vertex graph with spectral expansion λ. Then $\forall S,T \subseteq V$, we have*

$$\left| |E(S,T)| - \frac{d|S|\cdot|T|}{n} \right| \le \lambda d\sqrt{|S||T|}.$$

Let us consider the two terms in the left side: the size of $E(S,T)$ is the number of edges between two sets, and $d|S|\cdot|T|/n$ is the expected number of edges between S and T in a random graph with edge density d/n. So small λ implies that G is "more" random.

Proof. Let $\mathbf{1}_S, \mathbf{1}_T$ be the characteristic vectors of S and T. Expand these vectors in the orthonormal basis of eigenvectors $v_1,\cdots,v_n$, i. e. $\mathbf{1}_S = \sum_i \alpha_i v_i$, and $\mathbf{1}_T = \sum_i \beta_i v_i$. Then

$$|E(S,T)| = \mathbf{1}_S \cdot A \cdot \mathbf{1}_T = \left(\sum_i \alpha_i v_i\right) A \left(\sum_i \beta_i v_i\right) = \sum_i \lambda_i \alpha_i \beta_i,$$

where λ_is are eigenvalues of A. Since $\alpha_1 = \langle \mathbf{1}_S, \frac{1}{\sqrt{n}} \rangle = \frac{|S|}{\sqrt{n}}, \beta_1 = \frac{|T|}{\sqrt{n}}$ and $\lambda_1 = d$, then

$$|E(S,T)| = d \cdot \frac{|S| \cdot |T|}{n} + \sum_{i=2}^{n} \lambda_i \alpha_i \beta_i.$$

Thus

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \sum_{i=2}^{n} \lambda_i \alpha_i \beta_i \leq \lambda \cdot d \cdot \sum_{i=2}^{n} |\alpha_i \beta_i|$$

By Cauchy-Schwartz inequality, we have

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \lambda ||\mathbf{1}_S|| \cdot ||\mathbf{1}_T|| = \lambda \cdot d \cdot \sqrt{|S| \cdot |T|}.$$

□

Some remarks about the expander mixing lemma.

Lemma 3.4 (Converse of the Expander Mixing Lemma)**.** *[BL06] Let G be a d-regular graph and suppose that*

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \leq \theta d \sqrt{|S||T|}$$

holds for every two disjoint sets S, T and for some positive θ. Then $\lambda = \mathcal{O}(\theta(1 + \log(d/\theta)))$.

In the following, we use a three-tuple (n, d, λ) to represent an n-vertex d-regular graph with spectral expansion λ.

Corollary 3.5. *The size of the independent set for any (n, d, λ)-graph is at most λn.*

Proof. Let $T = S$. By Expander Mixing Lemma, we get $|S| \leq \lambda n$. □

Corollary 3.6. *For any (n, d, λ)-graph G, the chromatic number $\chi(G) \geq 1/\lambda$.*

Proof. Let $c : V \to \{1, \cdots, k\}$ be a coloring of G. Then for every $1 \leq i \leq k$, $c^{-1}(i)$ is an independent set. Since the size of every independent set is at most λn, so the chromatic number is at least $1/\lambda$. □

3 Cheeger's Inequality

Expander Mixing Lemma states that on graphs with good expansion, the graph's edges are *well distributed* and the spectral expansion of graphs are closely related to behavior of *any cut* in a graph.

Theorem 3.7 (Cheeger's Inequality)**.** *Let G be a d-regular graph and the eigenvalues of $\mathbf{A}(G)$ are $\lambda_1 \geq \ldots \geq \lambda_n$. Then*

$$\frac{d - \lambda_2}{2} \leq h(G) \leq \sqrt{2d(d - \lambda_2)}.$$

References

[AC88] N. Alon and F. R. K. Chung. Explicit construction of linear sized tolerant networks. *Discrete Mathematics*, 72(1-3):15–19, 1988.

[BL06] Y. Bilu and N. Linial. Lifts, discrepancy and nearly optimal spectral gap. *Combinatorica*, 26(5):495–519, 2006.