Lecture 4: Cheeger's Inequality

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1 Statement of Cheeger's Inequality

In this lecture we assume for simplicity that G is a d-regular graph. We shall work with the normalized adjacency matrix $\mathbf{M} = \frac{1}{d}\mathbf{A}$. The goal of this class is to prove Cheeger's inequality which establishes an interesting connection between $1-\lambda_2$ and the (normalized) edge expansion.

Definition 4.1 ((Normalized) Edge Expansion of a Regular Graph)**.** *The normalized edge expansion of a d-regular graph G is defined as:*

$$h(G) = \min_{S\colon |S|\leqslant|V|/2} \frac{|E(S, V\setminus S)|}{d|S|}.$$

Theorem 4.2 ([Alo86, SJ89])**.** *For any d-regular graph $G = (V,E)$,*

$$\frac{h^2}{2} \leqslant 1-\lambda_2 \leqslant 2h,$$

or equivalently,

$$(1-\lambda_2)/2 \leqslant h \leqslant \sqrt{2(1-\lambda_2)}.$$

2 Preparations

We start with a useful equality that will be applied later on.

Lemma 4.3.

$$\sum_{u,v\in V} \mathbf{M}_{u,v}\cdot(x_u-x_v)^2 = 2x^{\mathrm{T}}x - 2x^{\mathrm{T}}\mathbf{M}x.$$

Proof. Let us compute

$$\sum_{u,v\in V} \mathbf{M}_{u,v}\cdot(x_u-x_v)^2 = \sum_{u,v\in V} \mathbf{M}_{u,v}\cdot\left(x_u^2+x_v^2\right) - 2\sum_{u,v\in V}\mathbf{M}_{u,v}x_ux_v.$$

For the first term:

$$\sum_{u,v\in V} \mathbf{M}_{u,v}\cdot\left(x_u^2+x_v^2\right) = 2\sum_{u,v\in V}\mathbf{M}_{u,v}\cdot x_u^2 = 2\sum_{u\in V}x_u^2\sum_{v\in V}\mathbf{M}_{u,v} = 2x^{\mathrm{T}}x.$$

Further,

$$\sum_{u\in V}\sum_{v\in V}\mathbf{M}_{u,v}x_ux_v = \sum_{u\in V}x_u\cdot\sum_{v\in V}\mathbf{M}_{u,v}x_v = \sum_{u\in V}x_u\cdot(\mathbf{M}x)_u = x^{\mathrm{T}}\mathbf{M}x.$$

Therefore,

$$\sum_{u,v\in V} \mathbf{M}_{u,v}\cdot(x_u-x_v)^2 = 2x^{\mathrm{T}}x - 2x^{\mathrm{T}}\mathbf{M}x.$$

□

Lemma 4.4 (Courant-Fischer Formula for λ_1 and λ_2)**.** *Let $\lambda_1 = 1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ be the n eigenvalues of the normalized adjacency matrix $\mathbf{M} = \frac{1}{d} \cdot \mathbf{A}$. Then,*

$$\lambda_1 = \max_{x \neq \mathbf{0}} \frac{x^{\mathrm{T}} \mathbf{M} x}{x^{\mathrm{T}} x}, \qquad \lambda_2 = \max_{\substack{x \neq \mathbf{0} \\ x \perp \mathbf{1}}} \frac{x^{\mathrm{T}} \mathbf{M} x}{x^{\mathrm{T}} x}.$$

Moreover,

$$1 - \lambda_2 = \min_{\substack{x \neq \mathbf{0} \\ x \perp \mathbf{1}}} \frac{\sum_{u,v \in V} \mathbf{M}_{u,v} \cdot (x_u - x_v)^2}{2 x^{\mathrm{T}} x} = \min_{\substack{x \neq \mathbf{0} \\ x \perp \mathbf{1}}} \frac{\sum_{\{u,v\} \in E} (x_u - x_v)^2}{2 d x^{\mathrm{T}} x}$$

Proof. We begin with the first statement. Let $v_1, \ldots, v_n$ be an orthonormal basis of eigenvectors. Let x be any non-zero vector and express x in terms of this basis of eigenvectors:

$$x = \sum_{i=1}^{n} \alpha_i v_i,$$

where $\alpha_i = v_i^{\mathrm{T}} x$. It follows that

$$\begin{aligned} x^{\mathrm{T}} \mathbf{M} x &= \left(\sum_{i=1}^{n} \alpha_i v_i \right)^{\mathrm{T}} \mathbf{M} \left(\sum_{i=1}^{n} \alpha_i v_i \right) \\ &= \left(\sum_{i=1}^{n} \alpha_i v_i \right)^{\mathrm{T}} \left(\sum_{i=1}^{n} \alpha_i \lambda_i v_i \right) \\ &= \sum_{i=1}^{n} \alpha_i^2 \lambda_i. \end{aligned}$$

Similarly,

$$x^{\mathrm{T}} x = \left(\sum_{i=1}^{n} \alpha_i v_i \right)^{\mathrm{T}} \left(\sum_{i=1}^{n} \alpha_i v_i \right) = \sum_{i=1}^{n} \alpha_i^2.$$

Therefore,

$$\frac{x^{\mathrm{T}} \mathbf{M} x}{x^{\mathrm{T}} x} = \frac{\sum_{i=1}^{n} \alpha_i^2 \lambda_i}{\sum_{i=1}^{n} \alpha_i^2} \leqslant \frac{\lambda_1 \sum_{i=1}^{n} \alpha_i^2}{\sum_{i=1}^{n} \alpha_i^2} = \lambda_1, \tag{1}$$

so,

$$\lambda_1 \geqslant \max_{x \neq 0} \frac{x^{\mathrm{T}} \mathbf{M} x}{x^{\mathrm{T}} x}.$$

Moreover, if x is an eigenvector with eigenvalue one, then $x^{\mathrm{T}} \mathbf{M} x = x^{\mathrm{T}} x$ and equation (1) becomes an equality. Thus

$$\lambda_1 = \max_{x \neq 0} \frac{x^{\mathrm{T}} \mathbf{M} x}{x^{\mathrm{T}} x}.$$

Consider now any vector $x \neq 0$ with $x \perp 1$. Then, $\alpha_1 = 0$ and therefore,

$$\frac{x^{\mathrm{T}} \mathbf{M} x}{x^{\mathrm{T}} x} = \frac{\sum_{i=2}^{n} \alpha_i^2 \lambda_i}{\sum_{i=2}^{n} \alpha_i^2} \leqslant \frac{\lambda_2 \sum_{i=2}^{n} \alpha_i^2}{\sum_{i=2}^{n} \alpha_i^2} = \lambda_2,$$

and we have equality again if x is the eigenvector of λ_2, since then all $\alpha_i = 0$ except for $i = 2$. Therefore,

$$\lambda_2 = \max_{\substack{x \neq 0 \\ x \perp \mathbf{1}}} \frac{x^{\mathrm{T}} \mathbf{M} x}{x^{\mathrm{T}} x}.$$

For the second statement, we know by Lemma 4.3 that

$$\begin{aligned}\lambda_2 &= \max_{\substack{x \neq \mathbf{0} \\ x \perp v_1}} \frac{x^{\mathrm{T}} x - \frac{1}{2}\sum_{u,v \in V} \mathbf{M}_{u,v} \cdot (x_u - x_v)^2}{x^{\mathrm{T}} x} \\ &= 1 - \min_{\substack{x \neq \mathbf{0} \\ x \perp v_1}} \frac{\sum_{u,v \in V} \mathbf{M}_{u,v} \cdot (x_u - x_v)^2}{2 x^{\mathrm{T}} x},\end{aligned}$$

which gives the second statement of the lemma. □

3 Proof of Cheeger's Inequality

For the proof of Cheeger's inequality, we introduce another related expansion parameter, the so-called conductance:

$$\Phi(G) := \min_{\emptyset \subsetneq S \subsetneq V} \frac{|E(S, V \setminus S)|}{d|S| \cdot \frac{|V \setminus S|}{|V|}}.$$

Intuitively, the conductance measures how close a graph is to a random d-regular graph. The reason is that for any set S, we have $d|S|$ incident edges and the proportion of edges going to $|V \setminus S|$ is $|V \setminus S|/|V|$.

Our next claim is that:

$$h(G) \leqslant \Phi(G) \leqslant 2h(G).$$

The left inequality follows directly from the definition. For the right inequality, we first note that

$$h(G) = \min_{\emptyset \subsetneq S \subsetneq V} \frac{|E(S, V \setminus S)|}{d \min\{|S|, |V \setminus S|\}}.$$

Further, $\frac{|S| \cdot |V \setminus S|}{|V|} \geqslant \min\{|S|, |V \setminus S|\} \cdot \frac{1}{2}$ and hence $h(G) \geqslant \frac{1}{2}\Phi(G)$.

Theorem 4.5. *For any d-regular graph $G = (V, E)$,*

$$\frac{h^2}{2} \leqslant 1 - \lambda_2 \leqslant 2h,$$

or equivalently,

$$(1 - \lambda_2)/2 \leqslant h \leqslant \sqrt{2(1 - \lambda_2)}.$$

Proof. **Proof of $1 - \lambda_2 \leqslant 2h$.**

We will relate $\Phi(G)$ to λ_2. We first formulate $\Phi(G)$ as an integer minimization problem to find out that λ_2 is essentially the relaxation of that problem.

$$\begin{aligned}\Phi(G) &= \min_{x \in \{0,1\}^n,\ x \notin \{0^n, 1^n\}} \frac{\frac{1}{2}\sum_{u,v \in V} d \cdot \mathbf{M}_{u,v} \cdot |x_u - x_v|}{d(\sum_{u \in V} x_u)(n - \sum_{u \in V} x_u) \cdot \frac{1}{n}} \\ &= \min_{x \in \{0,1\}^n,\ x \notin \{0^n, 1^n\}} \frac{\sum_{u,v \in V} \mathbf{M}_{u,v} \cdot (x_u - x_v)^2}{2(\sum_{u \in V} x_u)(n - \sum_{u \in V} x_u) \cdot \frac{1}{n}}\end{aligned}$$

We now rewrite the denominator using the following equality:

$$2\left(\sum_{u\in V} x_u\right)\cdot\left(n-\sum_{u\in V} x_u\right) = 2n\sum_{u\in V} x_u - 2\sum_{u,v\in V} x_u x_v = 2n\sum_{u\in V} x_u^2 - 2\sum_{u,v\in V} x_u x_v = \sum_{u,v\in V}(x_u-x_v)^2.$$

so that

$$\begin{aligned}
\Phi(G) &= \min_{x\in\{0,1\}^n,\ x\notin\{0^n,1^n\}} \frac{\sum_{u,v\in V}\mathbf{M}_{u,v}\cdot(x_u-x_v)^2}{\frac{1}{n}\sum_{u,v\in V}(x_u-x_v)^2} \\
&\geqslant \min_{x\in\mathbb{R}^n,\ x\notin\{0^n,1^n\}} \frac{\sum_{u,v\in V}\mathbf{M}_{u,v}\cdot(x_u-x_v)^2}{\frac{1}{n}\sum_{u,v\in V}(x_u-x_v)^2} \\
&= \min_{x\in\mathbb{R}^n,\ x\notin\{0^n,1^n\},x\perp\mathbf{1}} \frac{\sum_{u,v\in V}\mathbf{M}_{u,v}\cdot(x_u-x_v)^2}{\frac{1}{n}\sum_{u,v\in V}(x_u-x_v)^2} && (x\rightsquigarrow x+\alpha\cdot\mathbf{1}) \\
&= \min_{x\in\mathbb{R}^n,\ x\notin\{0^n,1^n\},x\perp\mathbf{1}} \frac{\sum_{u,v\in V}\mathbf{M}_{u,v}\cdot(x_u-x_v)^2}{2x^{\mathrm{T}}x-2\sum_{u,v\in V}x_u x_v} \\
&= \min_{x\in\mathbb{R}^n,\ x\notin\{0^n,1^n\},x\perp\mathbf{1}} \frac{\sum_{u,v\in V}\mathbf{M}_{u,v}\cdot(x_u-x_v)^2}{2x^{\mathrm{T}}x} && \left(\sum_{u,v} x_u x_v = \sum_u x_u \sum_v x_v = 0\right) \\
&= 1-\lambda_2.
\end{aligned}$$

Proof of the other direction. Let x be the corresponding eigenvector to λ_2. Assume without loss of generality that at most $n/2$ entries of x are positive (otherwise we work with $-x$). Define a vector $y\in\mathbb{R}^n$ as

$$y_u = \max\{x_u, 0\}.$$

We now prove the following inequalities:

Claim 4.6. *1.* $1-\lambda_2 \geqslant \frac{\sum_{u,v\in V}\mathbf{M}_{u,v}\cdot(y_u-y_v)^2}{2y^{\mathrm{T}}y}$.

2. $\sum_{u,v\in V}\mathbf{M}_{u,v}\cdot(y_u-y_v)^2 \geqslant \frac{1}{4y^{\mathrm{T}}y}\cdot\left(\sum_{u,v\in V}\mathbf{M}_{u,v}\cdot\left|y_u^2-y_v^2\right|\right)^2$

3. $\sum_{u,v\in V}\mathbf{M}_{u,v}\cdot\left|y_u^2-y_v^2\right| \geqslant 2hy^{\mathrm{T}}y$.

Assuming that the three inequalities in the above claim hold, we can finish the proof as follows:

$$\begin{aligned}
1-\lambda_2 &\overset{\text{Claim 1}}{\geqslant} \frac{\sum_{u,v\in V}\mathbf{M}_{u,v}\cdot(y_u-y_v)^2}{2y^{\mathrm{T}}y} \\
&\overset{\text{Claim 2}}{\geqslant} \frac{\left(\sum_{u,v\in V}\mathbf{M}_{u,v}\cdot\left|y_u^2-y_v^2\right|\right)^2}{8(y^{\mathrm{T}}y)^2} \\
&\overset{\text{Claim 3}}{\geqslant} h^2/2.
\end{aligned}$$

The proof of the first and second inequality of the claim are relatively straightforward and can be found in the lecture notes. We only give the proof of the third one, which is the most interesting one, as the edge expansion comes into play.

The third inequality is the most technical one. Let $V = \{1, \ldots, n\}$ and order the vertices so that $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n$. Let $t \in \{1, \ldots, n\}$ be the largest index such that $y_k > 0$. Then,

$$\sum_{u,v \in V} \mathbf{M}_{u,v} \cdot \left|y_u^2 - y_v^2\right| = 2\sum_{i=1}^{t} \sum_{j=i+1}^{n} \mathbf{M}_{i,j} \cdot \left(y_i^2 - y_j^2\right)$$
$$= 2\sum_{k=1}^{t} \sum_{i \leqslant k} \sum_{j > k} \mathbf{M}_{i,j} \cdot \left(y_k^2 - y_{k+1}^2\right),$$

where the last equality holds since for every $i < j$,

$$\sum_{k=i}^{j-1} \mathbf{M}_{i,j} \cdot \left(y_k^2 - y_{k+1}^2\right) = \mathbf{M}_{i,j} \cdot \left(y_i^2 - y_j^2\right)$$

Now define for any $k \in \mathbb{N}$, $S_k := \{1, \ldots, k\}$. Then,

$$\sum_{k=1}^{t} \sum_{i \leqslant k} \sum_{j > k} \mathbf{M}_{i,j} \cdot \left(y_k^2 - y_{k+1}^2\right) = \sum_{k=1}^{t} \left(y_k^2 - y_{k+1}^2\right) \cdot \frac{1}{d} \cdot |E(S_k, V \setminus S_k)|.$$

Moreover, by definition of the expansion, we have

$$\sum_{k=1}^{t} \left(y_k^2 - y_{k+1}^2\right) \cdot \frac{1}{d} \cdot |E(S_k, V \setminus S_k)| \geqslant \sum_{k=1}^{t} hk \cdot \left(y_k^2 - y_{k+1}^2\right) \qquad \text{(since } t \leqslant n/2\text{)}$$
$$= h \cdot \sum_{k=1}^{t} k \cdot (y_k^2 - y_{k+1}^2)$$
$$= h \cdot \sum_{k=1}^{t} y_k^2$$
$$= hy^{\mathrm{T}} y.$$

□

The proof of the inequality $1 - \lambda_2 \geqslant h^2/2$ suggests the following algorithm for finding a small cut.

1. Compute the eigenvector x corresponding to the largest eigenvalue λ_2
2. Order the vertices so that $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$
3. Try all cuts of the form $(\{1, 2, \ldots, k\}, \{k+1, \ldots, n\})$ and return the smallest one

By the proof of Cheeger's inequality (Theorem 4.2), it follows that the cut returned by above algorithm is at most $\sqrt{2 - 2\lambda_2}$.

4 Example

We apply our algorithm to the following graph:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Clearly, the corresponding graph G is 3-regular.

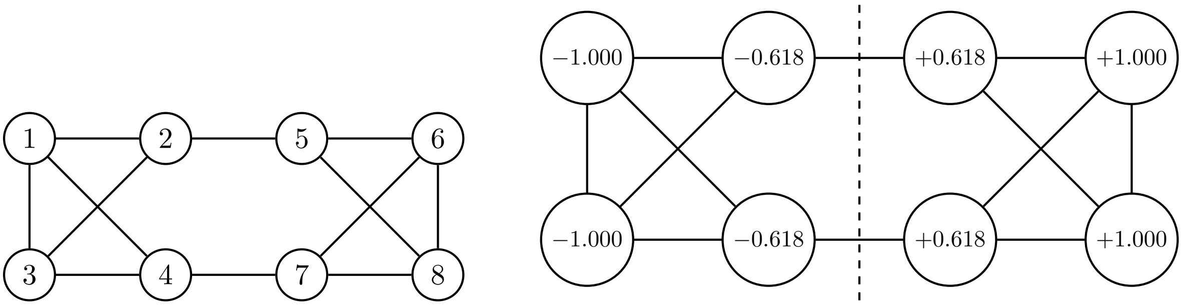

Figure 1: Illustration of the graph G. The labels of the nodes on the right hand side display the corresponding entries of the eigenvector of the second largest eigenvalue (which is $\sqrt{5}/3$). The dashed line describes the best possible cut, which is found by our algorithm.

5 Further Examples

Let us now apply Cheeger's inequality to three different graphs:

- Let G be a complete graph with n vertices. Then, $\lambda_2 = 0$ and

$$h(G) \leqslant \min_{S\colon |S| \leqslant |V|/2} \frac{|S| \cdot |V \setminus S|}{d|S|} = \min_{S\colon |S| \leqslant |V|/2} \frac{|V \setminus S|}{n-1} = \frac{n/2}{n-1} \approx 1/2.$$

- Let G be a cycle with n vertices. Any set $S \subseteq V$ with $1 \leqslant |S| \leqslant n/2$ has at least two edges from S to $V \setminus S$. Hence,

$$h(S) \geqslant \frac{2}{d|S|} \geqslant \frac{2}{2 \cdot n/2} = \frac{2}{n},$$

  where the inequality is tight if S is a set of consecutive vertices of size $n/2$. Further, $\lambda_2 = \cos(2\pi/n) \approx 1 - \Theta(1/n^2)$, so the spectral and geometric expansion differ by a square.

- For the hypercube G with $n = 2^d$ vertices, you are asked in an exercise to prove that $\lambda_2 = 1 - \frac{1}{d}$. Hence from Cheeger's inequality, $h \geqslant \frac{1-\lambda_2}{2} = \frac{1}{2d}$. Choosing S to be all vertices whose first bit equals zero yields

$$h \leqslant \frac{|S|}{d|S|} = \frac{1}{d}.$$

  Therefore, $h \sim 1/d$.

References

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