Lecture 8: Construction of Expanders

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In this lecture we study the explicit constructions of expander graphs. Although we can construct expanders probabilistically, this does not suffice for many applications.

- One applications of expander graphs is for reducing the randomness complexity of algorithms (cmp. Lecture 6), thus constructing the graph itself randomly does not serve this purpose.
- Sometimes we may even need expanders of exponential size. In this case, we cannot store the whole description of the graph.

Hence we need a more explicit version of expanders. We call a family of expander graphs *explicitly constructible* if the construction satisfies the following properties:

1. We can construct the whole graph in time poly(n), where n is the number of vertices in a graph.
2. For any vertex v and integer $i \in \{1, \cdots, d\}$, we can find the i-th neighbor of v in time poly($\log n, \log d$).
3. For any vertices v and u, we can determine if they are adjacent in time poly($\log n$).

Let $G = (V, E)$ be a d-regular graph. For each vertex $v \in V$, we label the edges adjacent to v and let $v[i]$ be the i-th edge of v. Define a *rotation map* $\mathrm{Rot}_G : V \times [d] \to V \times [d]$ by $\mathrm{Rot}_G(u, i) = (v, j)$ where v is the i-th neighbor of u and u is the j-th neighbor of v.

Example: For the graph shown in Figure 1, we have $\mathrm{Rot}(a, 3) = (h, 1)$.

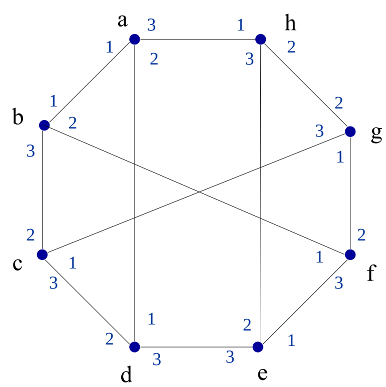

Figure 1: Rotation Map

1 Graph Powering

We know that if G is an (N, d, λ)-graph, then G^k, the *k-th powering* of the adjacency matrix of G, represents an (N, d^k, λ^k)-graph. From the rotation map's perspective, if G is a d-regular graph with rotation map Rot_G, then the k-th powering of G is a d^k-regular graph whose rotation map is given by $\mathrm{Rot}_{G^k}(v_0, (a_1, \cdots, a_k)) = (v_k, (b_k, \cdots, b_1))$, where the values $b_1, \cdots, b_k$ and v_k are computed via the rule $(v_i, b_i) = \mathrm{Rot}_G(v_{i-1}, a_i)$.

2 Replacement Product

For a D-regular graph G with N vertices and a d-regular graph H with D vertices, the replacement product, denoted as $G \circledR H$, is a $(d+1)$-regular graph with $N \cdot D$ vertices. Each vertex in G is replaced by a graph H, called *a cloud*. Moreover, $\mathrm{Rot}_{G \circledR H}((u, k), i) = ((v, \ell), j)$ if and only if $u = v$ and $\mathrm{Rot}_H(k, i) = (\ell, j)$, or $i = j = d + 1$ and $\mathrm{Rot}_G(u, k) = (v, \ell)$.

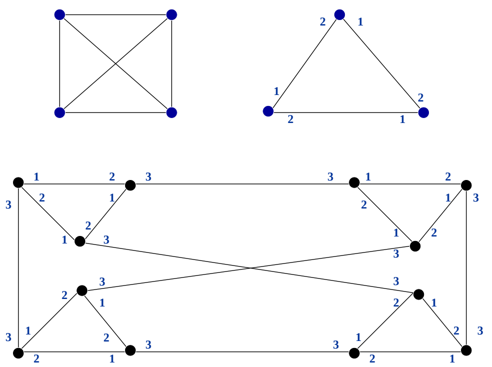

Figure 2: Replacement Product

Some comments:

- Replacement product depends on the arbitrary labels for the vertices and the edges.
- Replacement product reduces the (relative) vertex degree without losing the connectivity.
- For several problems in graph theory it can be shown that it suffices to solve them for graphs obtained by the Replacement product.

3 Zig-Zag Product

Based on rotation maps, the zig-zag product is defined as follows.

Definition 8.1. *[RVW00] Let G be a D-regular graph on $[N]$ with rotation map Rot_G and H be a d-regular graph on $[D]$ with rotation map Rot_H. Then their zig-zag product $G \circledz H$ is defined to be the d^2-regular graph on $[N] \times [D]$ whose rotation map $\mathrm{Rot}_{G \circledz H}$ is as follows:*

1. *Let $(a', i') = \mathrm{Rot}_H(a, i)$*
2. *Let $(w, b') = \mathrm{Rot}_G(v, a')$*
3. *Let $(b, j') = \mathrm{Rot}_H(b', j)$*
4. *Output $((w, b), (j', i'))$ as the value of $\mathrm{Rot}_{G \circledz H}((v, a), (i, j))$.*

Example: Let G and H be two graphs shown in Figure 3. Then $\mathrm{Rot}_{G \circledz H}((C, z), (1, 2)) = ((F, z), (1, 2))$.

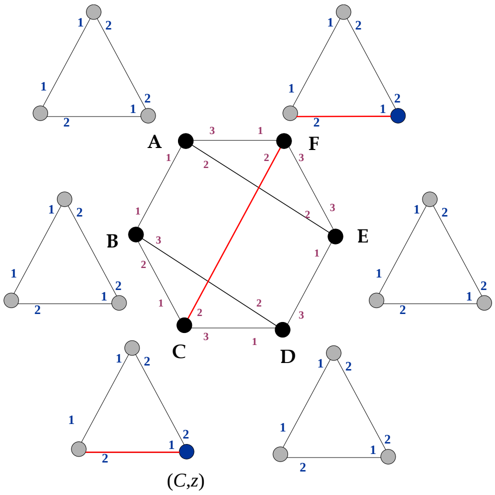

Figure 3: An example of the zig-zag product

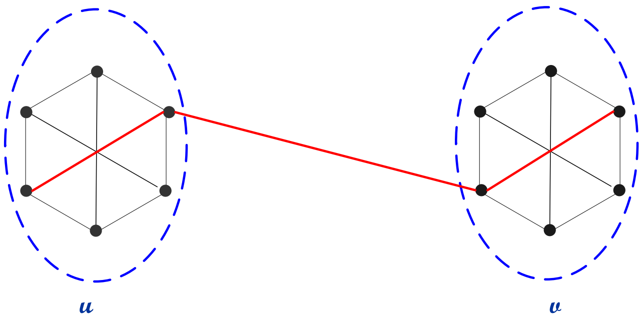

Figure 4: Intuition behind the zig-zag product

The intuition behind the zig-zag product is shown in Figure 4.

The zig-zag product corresponds to 3-step walks on the replacement product graph, where the first and the last steps are along the inner-cloud edges and the middle step is along an inter-cloud edge, and each vertex in the cloud corresponds to an edge starting from the vertex which the cloud represents.

Theorem 8.2. *Suppose that G is an (N_1, d_1, λ_1)-expander and H is a (d_1, d_2, λ_2)-expander. Then $G ⓩ H$ is an $(N_1 d_1, d_2^2, f(\lambda_1, \lambda_2))$-expander, where $f(\lambda_1, \lambda_2) \le \lambda_1 + \lambda_2 + \lambda_2^2$.*

Proof. The number of vertices and degree of $G ⓩ H$ are obtained directly from the definition of the zig-zag product and we only need to analyze the spectral expansion of $G ⓩ H$. Let M be the normalized adjacency matrix of $G ⓩ H$. It suffices to show that for any $\alpha \perp \mathbf{1}_{N_1 d_1}, \alpha \in \mathbb{R}^{N_1 d_1}$, it holds that

$$|\langle M\alpha, \alpha\rangle| \le f(\lambda_1, \lambda_2) \cdot |\langle \alpha, \alpha \rangle|.$$

Let $\alpha \in \mathbb{R}^{N_1 d_1}$ with the property that $\alpha \perp \mathbf{1}_{N_1 d_1}$. For any vertex $v \in [N_1]$, define $\alpha_v \in \mathbb{R}^{d_1}$ by $(\alpha_v)_k = \alpha_{vk}$. Also, let $C : \mathbb{R}^{N_1 d_1} \to \mathbb{R}^{N_1}$ be a linear mapping such that $(C\alpha)_v = \sum_{k=1}^{d_1} \alpha_{vk}$. Then we can express α as

$$\alpha = \sum_{v \in [N_1]} e_v \otimes \alpha_v.$$

Let $\alpha_v = \alpha_v^{\parallel} + \alpha_v^{\perp}$ where $\alpha_v^{\perp} \perp \mathbf{1}_{d_1}$. Then

$$\begin{aligned}\alpha &= \sum_v \left(e_v \otimes \alpha_v^{\parallel}\right) + \sum_v \left(e_v \otimes \alpha_v^{\perp}\right) \\ &:= \alpha^{\parallel} + \alpha^{\perp}.\end{aligned}$$

That is, $\alpha^{\parallel}$ is uniform within any given cloud and can be expressed as

$$\alpha^{\parallel} = \frac{C\alpha \otimes \mathbf{1}_{d_1}}{d_1}.$$

Let A and B be the normalized adjacency matrices of G and H, respectively. Let $\tilde{B} = I_{N_1} \otimes B$ and $\tilde{A}$ be the permutation matrix corresponding to Rot_G, i. e. an $N_1 d_1 \times N_1 d_1$ matrix where

$$\tilde{A}_{(u,i)(v,j)} = \begin{cases} 1 & \text{if } \mathrm{Rot}_G(u,i) = \mathrm{Rot}_G(v,j) \\ 0 & \text{otherwise.}\end{cases}$$

Then $M = \tilde{B}\tilde{A}\tilde{B}$. Since $\tilde{B}$ is real symmetric, we have

$$\langle M\alpha, \alpha\rangle = \langle \tilde{B}\tilde{A}\tilde{B}\alpha, \alpha\rangle = \langle \tilde{A}\tilde{B}\alpha, \tilde{B}\alpha\rangle.$$

On the other hand, we have $\tilde{B}\alpha = \tilde{B}(\alpha^{\parallel} + \alpha^{\perp}) = \alpha^{\parallel} + \tilde{B}\alpha^{\perp}$. Thus

$$\begin{aligned}\langle M\alpha, \alpha\rangle &= \left\langle \tilde{A}\left(\alpha^{\parallel} + \tilde{B}\alpha^{\perp}\right), \left(\alpha^{\parallel} + \tilde{B}\alpha^{\perp}\right)\right\rangle \\ &= \langle \tilde{A}\alpha^{\parallel}, \alpha^{\parallel}\rangle + \langle \tilde{A}\alpha^{\parallel}, \tilde{B}\alpha^{\perp}\rangle + \langle \tilde{A}\tilde{B}\alpha^{\perp}, \alpha^{\parallel}\rangle + \langle \tilde{A}\tilde{B}\alpha^{\perp}, \tilde{B}\alpha^{\perp}\rangle\end{aligned}$$

and

$$\begin{aligned}|\langle M\alpha, \alpha\rangle| &\le \left|\left\langle \tilde{A}\alpha^{\parallel}, \alpha^{\parallel}\right\rangle\right| + \|\tilde{A}\alpha^{\parallel}\| \cdot \|\tilde{B}\alpha^{\perp}\| + \|\tilde{A}\tilde{B}\alpha^{\perp}\| \cdot \|\alpha^{\parallel}\| + \|\tilde{A}\tilde{B}\alpha^{\perp}\| \cdot \|\tilde{B}\alpha^{\perp}\| \\ &= \left|\left\langle \tilde{A}\alpha^{\parallel}, \alpha^{\parallel}\right\rangle\right| + 2\|\alpha^{\parallel}\| \cdot \|\tilde{B}\alpha^{\perp}\| + \|\tilde{B}\alpha^{\perp}\|^2, \end{aligned} \tag{1}$$

where the last equality holds as $\tilde{A}$ is a permutation and $\|\tilde{A}x\| = \|x\|$ for any $x \in \mathbb{N}^{N_1 d_1}$.

Notice that

$$\begin{aligned}\|\tilde{B}\alpha^{\perp}\|^2 &= \left\|\tilde{B}\left(\sum_v e_v \otimes \alpha_v^{\perp}\right)\right\|^2 \\ &= \left\|\sum_v e_v \otimes B\alpha_v^{\perp}\right\|^2 \\ &= \sum_v \left\|B\alpha_v^{\perp}\right\|^2 \\ &\le \sum_v \lambda_2^2 \left\|\alpha_v^{\perp}\right\|^2 \\ &\le \lambda_2^2 \left\|\alpha^{\perp}\right\|^2. \end{aligned} \tag{2}$$

So we only need to bound $\left|\left\langle \tilde{A}\alpha^{\parallel}, \alpha^{\parallel}\right\rangle\right|$.

$$\begin{aligned}\left\langle \tilde{A}\alpha^{\parallel}, \alpha^{\parallel}\right\rangle &= \left\langle \tilde{A}\alpha^{\parallel}, C\alpha \otimes \mathbf{1}_{d_1}\right\rangle / d_1 \\ &= \left\langle C\tilde{A}\alpha^{\parallel}, C\alpha\right\rangle / d_1 \\ &= \langle AC\alpha, C\alpha\rangle / d_1\end{aligned}$$

and

$$\begin{aligned}\left|\left\langle \tilde{A}\alpha^{\|}, \alpha^{\|}\right\rangle\right| &\le \lambda_1 \langle C\alpha, C\alpha\rangle / d_1 \\ &= \lambda_1 \langle C\alpha \otimes \mathbf{1}_{d_1}, C\alpha \otimes \mathbf{1}_{d_1}\rangle / d_1^2 \\ &= \lambda_1 \cdot \langle \alpha^{\|}, \alpha^{\|}\rangle \\ &= \lambda_1 \left\|\alpha^{\|}\right\|^2 . \end{aligned} \tag{3}$$

Combining equation (2) and equation (3), we have

$$|\langle M\alpha, \alpha\rangle| \le \lambda_1 \left\|\alpha^{\|}\right\|^2 + 2\lambda_2 \left\|\alpha^{\|}\right\| \cdot \left\|\alpha^{\perp}\right\| + \lambda_2^2 \left\|\alpha^{\perp}\right\|^2 .$$

By taking $p = \frac{\|\alpha^{\|}\|}{\|\alpha\|}$ and $q = \frac{\|\alpha^{\perp}\|}{\|\alpha\|}$, we have $p^2 + q^2 = 1$. Therefore

$$\begin{aligned}\frac{|\langle M\alpha, \alpha\rangle|}{|\langle \alpha, \alpha\rangle|} &\le \lambda_1 p^2 + 2\lambda_2 pq + \lambda_2^2 q^2 \\ &\le \lambda_1 + \lambda_2 + \lambda_2^2,\end{aligned}$$

which completes the proof. □

Some comments on the zig-zag product.

- The edge labels in $G \textcircled{z} H$ are just pairs of edges labeled in the small graph.
- By taking a product of a large graph with a small graph, the resulting graph inherits (roughly) its size from the large one, its degree from the small one and its expansion properties from both. This was the key to creating arbitrary large graphs with bounded degree.

4 Construction of Expanders

In this section we use the zig-zag product to construct expander graphs.

Theorem 8.3. *Let H be a (d^4, d, λ_0) graph for some $\lambda_0 \le 1/5$. Define $G_1 = H^2$ and $G_{t+1} = G_t^2 \textcircled{z} H$ for $t \ge 1$. Then for all t, G_t is a (d^{4t}, d^2, λ)-expander with $\lambda \le 2/5$.*

Proof. We prove the theorem by induction. When $t = 1$, it is straightforward to see that G_1 is a (d^4, d^2, λ_0^2)-expander. Assume that G_{t-1} is a $(d^{4(t-1)}, d^2, \lambda)$-expander for $\lambda \le 2/5$. By Definition 8.1, the number of vertices in G_t is d^{4t}. So it suffices to show the spectral expansion of G_t. By Theorem 8.2, the spectral expansion of G_t is

$$\begin{aligned}\lambda(G_t) &\le \lambda(G_{t-1}^2) + \lambda(H) + \lambda(H^2) \\ &= \left(\frac{2}{5}\right)^2 + \frac{1}{5} + \frac{1}{25} \\ &= \frac{2}{5}.\end{aligned}$$

□

Remark. Since H is a graph of constant-size, we can find it in constant time by brute-force search.

Generalization.

References

[RVW00] Omer Reingold, Salil P. Vadhan, and Avi Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders and extractors. In *FOCS*, pages 3–13, 2000.