Background Knowledge of the Course

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1 Algebra

Eigenvalues. Given a matrix $\mathbf{A}$, a vector $x \neq 0$ is defined to be an eigenvector of $\mathbf{A}$ if and only if there is a $\lambda \in \mathbb{C}$ such that $\mathbf{A}x = \lambda x$. In this case, λ is called an eigenvalue of $\mathbf{A}$.

- A matrix $\mathbf{A}$ is called a Hermitian matrix if $\mathbf{A}_{i,j} = \overline{\mathbf{A}_{j,i}}$ for any element $\mathbf{A}_{i,j}$. Hermitian matrices always have real eigenvalues.
- **(Schur)** Let $\mathbf{A}$ be a real symmetric matrix with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Then $\sum_{i=1}^{t} \mathbf{A}_{i,i} \leq \sum_{i=1}^{t} \lambda_i$ for $1 \leq t \leq n$.
- **Courant-Fischer Formula**. Let $\mathbf{A}$ be an n by n symmetric matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and corresponding eigenvectors $v_1, \ldots, v_n$. Then

$$\lambda_1 = \min_{\|x\|=1} x^{\mathrm{T}}\mathbf{A}x = \min_{x \neq \mathbf{0}} \frac{x^{\mathrm{T}}\mathbf{A}x}{x^{\mathrm{T}} \cdot x},$$

$$\lambda_2 = \min_{\substack{\|x\|=1 \\ x \perp v_1}} x^{\mathrm{T}}\mathbf{A}x = \min_{\substack{x \neq \mathbf{0} \\ x \perp v_1}} \frac{x^{\mathrm{T}}\mathbf{A}x}{x^{\mathrm{T}} \cdot x},$$

$$\lambda_n = \max_{\|x\|=1} x^{\mathrm{T}}\mathbf{A}x = \max_{x \neq \mathbf{0}} \frac{x^{\mathrm{T}}\mathbf{A}x}{x^{\mathrm{T}}x}.$$

Trace. For any matrix $\mathbf{A}$, the trace of $\mathbf{A}$ is defined as $\mathrm{tr}(\mathbf{A}) := \sum_{i=1}^{n} \mathbf{A}_{i,i}$.

- For any matrix $\mathbf{A}$, the trace of $\mathbf{A}$ equals the sum of all the eigenvalues of $\mathbf{A}$.
- For any matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathrm{tr}(\mathbf{A}+\mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B})$.
- For any matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$.
- For any $k \in \mathbb{R}$, $\mathrm{tr}(k\mathbf{A}) = k \cdot \mathrm{tr}(\mathbf{A})$.

Inner Product.

- For any symmetric matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $x, y \in \mathbb{R}^n$, it holds that $\langle \mathbf{A}x, y \rangle = \langle x, \mathbf{A}y \rangle$.
- **Cauchy-Schwarz Inequality.** For any vectors x and y, it holds that

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|.$$

Tensor Product. For vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, their tensor product is the vector $x \otimes y \in \mathbb{R}^{n \times m}$ whose (i,j)-th entry is $x_i \cdot y_j$.

Norm. For any vector $x \in \mathbb{R}^n$, let $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$ for $1 \le p < \infty$. In particular, let $\|x\|_\infty := \max_i |x_i|$.

- **Norm Inequality.** For any $1 \le p \le q < \infty$ and $x \in \mathbb{R}^n$, it holds that

$$\|x\|_q \le \|x\|_p \le n^{1/p-1/q} \cdot \|x\|_q.$$

- **Hölder Inequality.** For any p, q with $1/p + 1/q = 1$, it holds that

$$\sum |x_i y_i| \le \left(\sum |x_i|^p\right)^{1/p} \cdot \left(\sum |y_i|^q\right)^{1/q}.$$

- **Minkowski's Inequality.** For any vectors x, y and $p \in \mathbb{R}$, it holds that $\|x+y\|_p \le \|x\|_p + \|y\|_p$.
- **Pythagorean Theorem.** For any two orthogonal vectors x and y we have $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

2 Complexity Theory

Complexity Class RP. The complexity class **RP** consists of all languages L for which there exists a probabilistic polynomial-time Turing machine M such that

$$\begin{aligned} m \in L &\Longrightarrow \Pr[M(x) = 1] \ge \frac{3}{4}, \\ m \notin L &\Longrightarrow \Pr[M(x) = 1] = 0. \end{aligned}$$

Complexity Class BPP. The complexity class **BPP** consists of all languages L for which there exists a probabilistic polynomial-time Turing machine M such that

$$\begin{aligned} m \in L &\Longrightarrow \Pr[M(x) = 1] \ge \frac{3}{4}, \\ m \notin L &\Longrightarrow \Pr[M(x) = 0] \ge \frac{3}{4}. \end{aligned}$$

It is straightforward to see that $\mathbf{P} \subseteq \mathbf{RP} \subseteq \mathbf{BPP}$.

Complexity Class L. The complexity class **L** consists of all languages decidable within deterministic logarithmic space.