Exercises for Algorithmic Game Theory: Assignment 11
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Problem 1. Consider a n-player game, where the set of pure strategies available to player i is given by S_i. Let $S = S_1 \times \cdots \times S_n$ denote the set of all strategy-profiles. Recall that any correlated equilibrium can be represented as a probability distribution over support S. Show that any convex combination of two such correlated equilibria will also be a correlated equilibrium.

Problem 2. Consider the following game. The players correspond to the nodes in a undirected graph $G = (V, E)$. Each node has to decide whether to color itself *black* or *white*. A strategy-profile in this game is represented as $\vec{s} = (\vec{s}_1, \ldots, \vec{s}_{|V|})$, where $\vec{s}_i \in \{\text{black, white}\}$ denotes the color chosen by node $i \in V$. The utility of node $i \in V$ is defined as follows.

$$u_i(\vec{s}) = \begin{cases} 2 & \text{if } \vec{s}_i = \text{ black;} \\ 1 + \left|\left\{j \in V : (i,j) \in E \text{ and } \vec{s}_j = \text{ black}\right\}\right| & \text{if } \vec{s}_i = \text{ white.} \end{cases}$$

Thus, the utility of a black node is 2; and the utility of a white node is 1 plus the number of its black neighbors. Every node wants to maximize its own utility. The price of anarchy is defined in terms of the sum of the utilities obtained by all the nodes.

(a). Show that the price of anarchy (of pure Nash equilibria) in this game can be as bad as $\Omega(|V|)$. (You should specify the graph and the assignment of colors in order to show this lower bound.)

(b). Use the smoothness argument to show that the price of anarchy (of correlated equilibria) in this game is upper bounded by $O(|V|)$.

(c). Show that this game always admits a pure Nash equilibrium.

Hint: Consider the potential function described below.

$$\Phi(\vec{s}) = |\{i \in V : \vec{s}_i = \text{ white}\}| + |\{(i,j) \in E : \vec{s}_i = \vec{s}_j = \text{ black}\}|.$$

Problem 3. We have a social network represented by an undirected graph $G = (V, E)$. Each node $i \in V$ in this graph denotes a person. There is a undirected edge $(i, j) \in E$ ifff i and j are friends. Each person $i \in V$ has an intrinsic *opinion*, which can be measured as a real number s_i. However, a person can be influenced by her friends to change her opinion. To be more specific, consider the game described below.

The strategy of a person $i \in V$ consists of deciding her final opinion z_i (which can be different from s_i). The cost incurred by person $i \in V$ is given by the following expression.

$$(z_i - s_i)^2 + \sum_{j \in V : (i,j) \in E} (z_i - z_j)^2$$

The first term measures the cost of diverging from her own intrinsic opinion. The second term measures the cost of diverging from the opinion of her friends. Every node (i.e., person) wants to minimize her own cost. The price of anarchy of this game is defined in terms of the sum of the costs incurred by all the nodes.

Show that the price of anarchy of pure Nash equilibria in this game is at most 2.