Summary of "Kuratowski's Theorem and Whitney's Duality"

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Abstract

In this summary, we give the short proof about Kuratowski's theorem which states the theorem of planar graph existing conditions. A planar graph excludes some forbidden structures as the subdivision of K_5 or $K_{3,3}$. We prove Kuratowski's Theorem based on 3-connected graph case firstly, then the proof is extended to general graph. Furthermore, we introduce the concept and relation about combinatorial duals and geometric dual. Finally Whitney' duality for the 2-connected multi-graph which illustrates the connection between planar embedding and duality has been given with short proof.

1 Planar Graph

we explain our proof on basis of the multi-graph, which allows distinct edges to have the same pair of end-vertices in a general graph. In this case, such edges may be parallel and form multiple edges. Hereinafter, when we use graph which also could be meaning of multi-graph, we don't distinguish them very precisely below. The minimum cut of a graph is a set of edges E of a graph G and $G-E$ will disconnect two non-empty disjoint sets of vertices, meanwhile you cannot find another set of edges with less cardinality to do this.

Simply, a graph is planar graph if it can be drawn on the plane in such a way that no edges intersect. Any cycle in the graph that surrounds a region without any edges reaching from the cycle into the region forms a face. In a planar graph, there always exists an external or unbounded face as the outer face. Euler's formula states that if a finite, connected, planar graph must have $v-e+f=2$, here v is the number of vertices, e is the number of edges and f is the number of faces. However, a more elegant characterization of planar graphs in terms of forbidden graphs was provided by the Polish mathematician Kazimierz Kuratowski, now known as *Kuratowski's theorem.* The complete graph K_5 Figure 3.1 and the bipartite graph $K_{3,3}$ Figure 3.2 are these forbidden graphs for planar graphs. A *subdivision* of a graph G is a graph resulting from an addition of a new vertex between two vertices and the replacement of the edge with two new edges.

2 Kuratowski's Theorem

After introducing above concepts, now we come to our theorem and proof part.

Theorem 1 (Kuratowski's theorem)**.** *A graph is planar if and only if it does not contain a subdivision of K_5 or a subdivision of $K_{3,3}$ as a subgraph.*

Lemma 1. *Every 3-connected graph of order at least five contains an edge e such that the graph $G//e$ is 3-connected graph.*

The forbidden subgraphs K_5 and $K_{3,3}$ are called *Kuratowski's graphs*. It is easy to find that $K_{3,3}$ and K_5 are nonplanar, and also their subdivisions cannot be represented in the plane. Due to this fact, the "easy part" of Kuratowski's theorem has been proved. Now we focus on the proof of converse part. We have to derive some new results about planar graphs in order to proceed in our proof. A graph G is *straight line embedded* if each edge is a straight line segment. If each bounded face of a straight embedded graph is convex, and the unbounded face is a complement of a convex set, then the embedding of G is said to be *convex*. We will prove Kuratowski's theorem for 3-connected graphs, which states a 3-connected graph can have a convex embedding in the plane then it is a planar embedding. The proof will be given in induction way, and we increase the number of vertices by induction. we have to use Lemma 1 in our proof, which requests a operation $G//e$. Here G/e is the graph obtained from G by edge-contracted on edge e, then $G//e$ is a graph generated from edge-contracted graph G/e by replacing all multiple edges by single edges joining the same pairs of vertices.

Lemma 2. *If G is a 3-connected graph with no subdivision of $K_{3,3}$ and K_5 as a subgraph, then G has a convex embedding in the plane.*

Proof. Our proof is given by induction on $n = |V(G)|$. It is trivial to verify the cases when $n = 4$ and 5. We continue our induction step by assuming that $n \geq 6$.

Since it is a 3-connected graph, by our Lemma 1, graph G must contain $e = xy$ such that $G' = G//e$ is 3-connected. Let z be the vertex in G' obtained by identifying x and y, see example in Figure 3.3. If G' contains a subdivision of forbidden graphs, then such a forbidden graph must be observed in G as well. Accordingly, we can assume G' is a convex embedded plane graph. As G' is 3-connected, then $G' - z$ is 2-connected graph. Thus, $G' - z$ containing the vertex z is bounded by a cycle C of G', and this cycle also can be found in graph G as Figure 3.4. The neighbors of x , $x_1, \ldots, x_2$ occurring on C are in the cyclic order. P_i is the path of C joining x_i and x_{i+1}, no any other x_j intersects them for example in Figure 3.7. If the neighbors of y only belong to a single path P_i then the graph G has a convex embedding referring to Figure 3.5 3.6. On the other hand, if this is not the case, then either the neighbors of y span more than one path in which case C together with x and y determines a subdivision of K_5 in Figure 3.8, or else the neighbors of y alternate with the neighbors of x on cycle C. In the later case, C together with x and y determine a subdivision of $K_{3,3}$ in Figure 3.9. So the other hand contradict with our assumption. Thus we always can have a convex embedding for a 3-connected graph without subdivision of K_5 or $K_{3,3}$. This completes our proof. □

After proving Lemma 2, we are going to extend our proof from 3-connected graphs to general graphs. Lemma 3 gives the concept of maximal planar. Suppose given any general graph G without containing subdivision K_5 or $K_{3,3}$. We continue connecting the non-adjacent vertices by adding new edges in G. After adding some new edges, the new graph G_1 without containing subdivision K_5 or $K_{3,3}$ is generated but any one more new edge addition in G_1 will obtain a forbidden subgraphs. Now our G_1 is maximal planar graph, which is 3-connected graph according to Lemma 3. Certainly by Lemma 2, this 3-connected graph G_1 must have a convex embedding in the plane. Furthermore deleting all these new adding edges will not change the convex embedding in the plane. Consequently, the general graph G have the convex embedding in that plane, which completes the proof for general graph.

Lemma 3. *If a graph G of order $>= 4$ contains no subdivision of K_5 or $K_{3,3}$ and the addition of any out of every the possible edges makes the graph non planar, then G is 3-connected.*

3 Whitney's Duality

We will introduce the concepts of combinatorial dual of graph and geometric dual graph. Then we introduce a important theorem in 2-connected graph, which illustrates the connection between planar graph and duality.

Proposition 1. *If G^* is a combinatorial dual of G and $E \subseteq E(G)$ is a set of edges of G such that $G - E$ has only one component containing edges, then G^*/E^* is a combinatorial dual of $G - E$.*

For a 2-connected plane multigraph G, then we define the geometric dual H of G as a plane multigraph that has precisely one vertex in each face of G. If e is an edge of G, then H has an edge e^* crossing the edge e from G and joining the two vertices of H in the two faces of G that contain e on the boundary. Now, we have following properties between original graph and corresponding dual graph.

1. If $E \subseteq E(G)$ is the edge set of a cycle in G, then E^* is a cut in H.
2. If E is the edge set of a forest in G, then $H - E^*$ is connected.

Proposition 2. *Let G be a 2-connected plane multigraph, and let H be its geometric dual. Then H is a combinatorial dual of G. Moreover, G is a geometric dual graph (and hence a combinatorial dual) of H.*

Here, we come to Whitney Theorem proof.

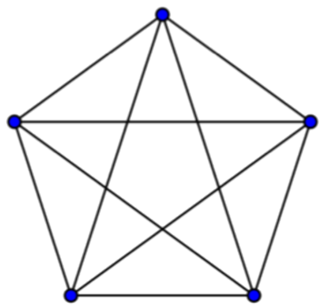

Figure 3.1: Graph K_5.

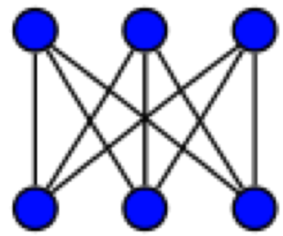

Figure 3.2: Graph $K_{3,3}$.

Theorem 2 (Whitney's Duality)**.** *Let G be a 2-connected multigraph, then G is planar if and only if it has a combinatorial dual. If G^* is a combinatorial dual of G, than G has an embedding in the plane such that G^* is isomorphic to the geometric dual of G. In particular, also G^* is planar, and G is a combinatorial dual of G^*.*

Proof. by Proposition 2, we only need to prove the second part of the theorem. We give proof by induction on the number of edges of G. If G is a simple cycle dividing the plane in two faces, any two edges of G^* are in a 2-cycle , therefore G^* has only two vertices. Clearly, G and G^* can be represented as a geometric dual pair. If G is not a cycle, then we can represent any non-cyclic G with a cycle G' and a path P connecting two vertices of G'. By the induction hypothesis, $H = G^*/E(P)^*$ is a combinatorial dual of G'. G' and H are geometric dual pair, and G' is also a combinatorial dual of H. If e_1, e_2 are two edges of P, then e_1^*, e_2^* are two edges of G^* which belong to a cycle C^* of G^*. Since P is not a cycle the correspondent set of edges E^* cannot be a minimal cut, so the all the edges $E(P)^*$ of the G^* must be parallel in G^* and join two vertices z_1, z_2 in G^*. The edges incident to the vertex z_0 form a minimal cut in G^*, therefore G' is a cycle separating the vertex z_0 from $H - z_0$. We can draw P inside G' defining two cycles C_1 and C_2 respectively containing subset of edges E_1 and E_2, that corresponds to E_1^* and E_2^* defining a cut for z_1 and z_2. This way we obtain a representation of G^* as a geometric dual of G. Finally, the combinatorial dual implies an embedding in the plane of G, this complete the proof. $\square$

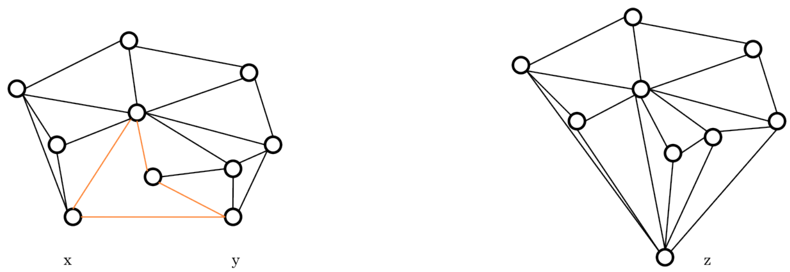

Figure 3.3: Example of identifying x and y

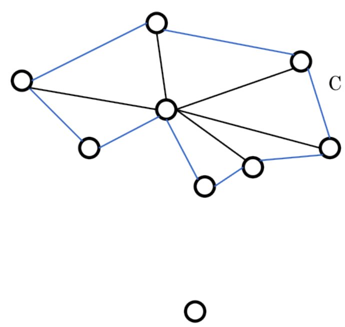

Figure 3.4: The cycle in $G' - z$

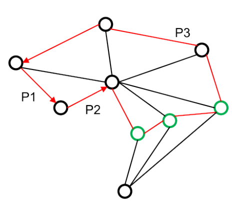

Figure 3.5: The neighbors of y.

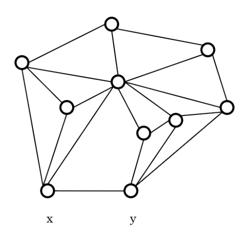

Figure 3.6: Convex embedding of G.

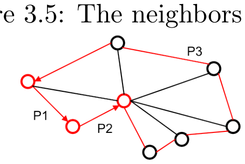

Figure 3.7: Paths of neighbors of x.

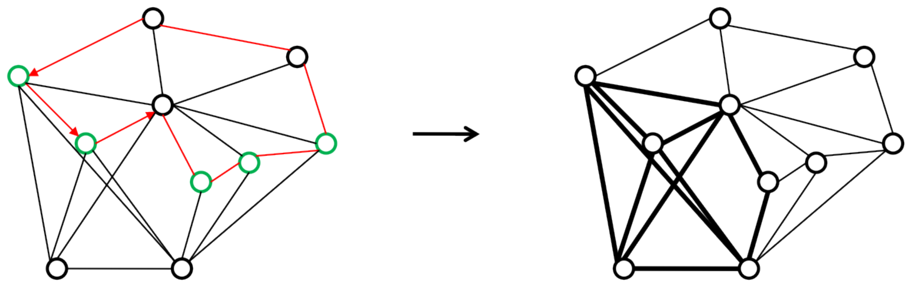

Figure 3.8: The neighbors of y in more than one path

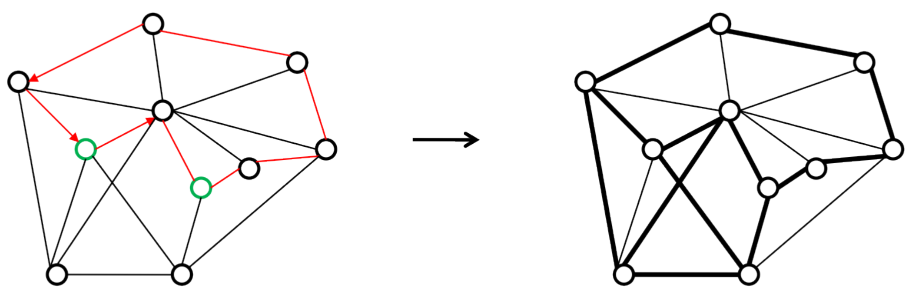

Figure 3.9: The neighbors of y alternate with the neighbors of x