Summary of ”Fast Subexponential Algorithm for Non-local Problems on Graphs of Bounded Genus” (Dorn, Fomin, Thilikos)

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Abstract

A planarization technique applied to embeddings of graphs on surfaces of bounded genus makes it possible to use a fast subexponential algorithm based on Dynamic Programming on the planarized graph. The result of these computations then can be used to find solutions for non-local hard graph problems in the original graph. As an example this general framework is applied to HAMILTONIAN-CYCLE for torus-embedded graphs. It provides us with subexponential algorithms for a wide class of computationally hard problems in graphs embedded on surfaces.

Some computationally hard graph problems have subexponential-time solutions obtainable with Dynamic Programming and Sphere-Cut-Decompositions, if the processed graphs are planar.

A Σ-embedded graph G is a graph that can be drawn on a compact, connected 2-manifold Σ such that edges only meet at vertices.

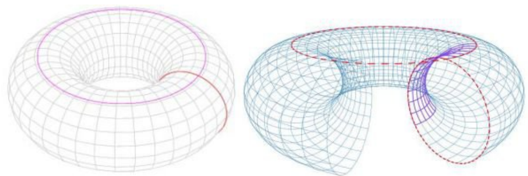
Planarity means that the graph is $\mathbb{S}_0$-embedded on the surface of the sphere.

Dynamic Programming is a fast problem solving technique for combinatorial optimization problems.

A **Sphere-Cut-Decomposition** is a special Branch-Decomposition of an $\mathbb{S}_0$-embedded graph G such that the vertices of each middle set define a tight noose on $\mathbb{S}_0$.

A **Branch-Decomposition** of a graph G is a ternary tree T where there exists a bijection between the edges of G and the leaves of T. Cutting the edges of T therefore partitions the edges of G into two disjoint sets, which have common incident vertices. This set of common vertices is called the **middle set** of the edge of T that generated it.

A **noose** on the sphere $\mathbb{S}_0$ or on the torus $\mathbb{S}_1$ is a subset of the surface of the sphere or a subset of the surface of the torus that is both an O-arc and G-normal. An **O-arc** is a subset of the surface of the sphere/torus that is homeomorphic to a circle. A subset of the surface of the sphere/torus is called **G-normal** if it meets an $\mathbb{S}_0$-embedded graph/an $\mathbb{S}_1$-embedded graph only at points that are vertices.



The left diagram shows grid graph embedded on a torus with two nonhomotopic noncontractible cycles. On the right, we see the grid after cutting along one such cycle. Note that this surface is a cylinder and the graph is now planar.

Figure 1: Cutting along a noncontractible tight noose of a $\mathbb{S}_1$-embedded graph

A noose that visits any region defined by the embedding only once is called **a tight noose**. A noose that is null-homotopic is called **a contractible noose**.

Definition 1 (**Cutting along a noose** N)**.** Let G' be the graph obtained from G by replacing N with two copies of N such that all edges on the left side of N incident to N are now incident to one copy of N and all edges on the right side of N are incident with the other copy of N. We say that G' is obtained from G by cutting along N. The copies N_X and N_Y of N are called **cut-nooses**.

Proposition 1. *Let G be a torus-embedded graph, let G' be the graph obtained from G by cutting along a noncontractible tight noose N on G. Then G' is planar. (cf. Figure 1)* [1]

Definition 2. HAMILTONIAN-CYCLE-Problem: Given a graph G, is there a cycle that visits each vertex exactly once?
HAMILTOR: HAMILTONIAN-CYCLE-Problem for a torus-embedded graph G.

Since a cut as shown in Figure 1 is along vertices, a possible Hamiltonian cycle in G will be disconnected.

Definition 3. A cut of a Hamiltonian cycle C in G along a tight noose N is a set of disjoint paths in G'.

After identifying the components of a Hamiltonian cycle in the planar graph G' they can be used to reconstruct the Hamiltonian cycle in G. Let N_X and N_Y be the cut-nooses obtained from cutting G along N. Also let $x_i \in N_X$ and $y_i \in N_Y$ be duplicated vertices of the same vertex in N.

Definition 4 (**Relaxed Hamiltonian Set of Paths**)**.** We call a set of disjoint paths P in G' relaxed Hamiltonian if:

1. Every path has its endpoints in N_X and N_Y.

2. Vertex x_i is an endpoint of some path P_i iff y_i is an endpoint of a path $P' \neq P$.

[1]Figure 1 from Frederic Dorn, 'Designing Subexponential Algorithms: Problems, Techniques and Structures', Thesis for the degree of Philosophiae Doctor (PhD), Department of Informatics, UNIVERSITY OF BERGEN, Bergen, Norway July 2007, p.18

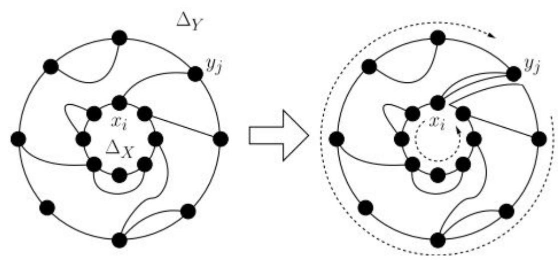


Figure 2: Combining two cut-nooses

3. If one of x_i ,y_i is an inner vertex of a path, the other one is not in any path.
4. Every vertex of $G^{'} \setminus N_X \cup N_Y$ is in some path.

A cut of a Hamiltonian cycle in G must have a corresponding relaxed Hamiltonian set of paths in $G^{'}$, but not vice versa. Checking in linear time for a single relaxed Hamiltonian set if it is also a solution for HAMILTOR is possible, by identifying the corresponding vertices of N_X and N_Y. From this follows that checking all possible relaxed Hamiltonian sets in $G^{'}$ will answer the question, whether a Hamiltonian cycle exists in G or not, and will return one, in case of existence.
But only equivalence classes on the set $HS(G^{'})$ of all of relaxed Hamiltonian sets have to be checked:

Definition 5. For any two sets $P_1, P_2 \in HS(G^{'})$, $P_1 \sim P_2$ if for every path in P_i there is a path in P_j with the same endpoints $(1 \leq i, j \leq 2)$.

Lemma 1. *The number of* ***different*** *equivalence classes of relaxed Hamiltonian sets in* $G^{'}$ *is in* $O(2^{3k})$ *where* k *is the length of the tight noose* N*.*

The proof of Lemma 1 is based upon the fact that the number of non-crossing paths with its endpoints in **one** noose corresponds in a way to the Catalan numbers. The two cut-nooses here are melted into one cut-noose by looking at one path in the relaxed Hamiltonian set and cutting the sphere along this path, cf. Figure 2. [2]

For reconstructing a possible Hamiltonian cycle, we start with a set of vertex tuples in $G^{'}$ called a candidate $K = \{(s_1, t_1), ..., (s_k, t_k)\}$ with $s_i, t_i \in N_X \cup N_Y, i = 1, ..., k$ and a vertex set $I \subset N_X \cup N_Y$. If there exists a relaxed Hamiltonian set P such that every (s_i, t_i) marks the endpoints of a path and the vertices of I are inner vertices of some paths, we can rebuild a Hamiltonian cycle in G from P. By using dynamic programming on Sphere-Cut-Decompostions it is checked for every candidate if there is a spanning subgraph of $G^{'}$ isomorphic to a Hamiltonian set $P \sim K$.

The framework above can - *after some careful modifications* - also be used for solving non-local problems on graphs **of bounded genus**. The main result is that in all cases the resulting algorithm has running time $2^{O(\sqrt{n})}$.

[2]Figure 2 from Frederic Dorn, Fedor V. Fomin, Dimitrios M. Thilikos: 'Fast subexponential algorithm for non-local problems on graphs of bounded genus', p.6

Parameterized problems in graphs embedded in surfaces of bounded genus can also be solved with this technique with running time $2^{O(\sqrt{p})} \cdot n^{O(1)}$ (for example: finding a cycle of length p, if it exists).