Summary of
"Fast Subexponential Algorithm for Non-local Problems on Graphs of Bounded Genus"
(Dorn, Fomin, Thilikos)

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Abstract

A planarization technique applied to embeddings of graphs on surfaces of bounded genus makes it possible to use a fast subexponential algorithm based on Dynamic Programming on the planarized graph. The result of these computations then can be used to find solutions for non-local hard graph problems in the original graph. As an example this general framework is applied to HAMILTONIAN-CYCLE for torus-embedded graphs. It provides us with subexponential algorithms for a wide class of computationally hard problems in graphs embedded on surfaces.

Some computationally hard graph problems have subexponential-time solutions obtainable with Dynamic Programming and Sphere-Cut-Decompositions, if the processed graphs are planar.

A $\Sigma$-embedded graph $G$ is a graph that can be drawn on a compact, connected 2-manifold $\Sigma$ such that edges only meet at vertices.

**Planarity** means that the graph is $\mathbb{S}_0$-embedded on the surface of the sphere.

**Dynamic Programming** is a fast problem solving technique for combinatorial optimization problems.

A **Sphere-Cut-Decomposition** is a special Branch-Decomposition of an $\mathbb{S}_0$-embedded graph $G$ such that the vertices of each middle set define a tight noose on $\mathbb{S}_0$.

A **Branch-Decomposition** of a graph $G$ is a ternary tree $T$ where there exists a bijection between the edges of $G$ and the leaves of $T$. Cutting the edges of $T$ therefore partitions the edges of $G$ into two disjoint sets, which have common incident vertices. This set of common vertices is called the **middle set** of the edge of $T$ that generated it.

A **noose** on the sphere $\mathbb{S}_0$ or on the torus $\mathbb{S}_1$ is a subset of the surface of the sphere or a subset of the surface of the torus that is both an O-arc and G-normal. An **O-arc** is a subset of the surface of the sphere/torus that is homeomorphic to a circle. A subset of the surface of the sphere/torus is called **G-normal** if it meets an $\mathbb{S}_0$-embedded graph/an $\mathbb{S}_1$-embedded graph only at points that are vertices.
A noose that visits any region defined by the embedding only once is called a **tight noose**. A noose that is null-homotopic is called a **contractible noose**.

**Definition 1 (Cutting along a noose)** $N$. Let $G'$ be the graph obtained from $G$ by replacing $N$ with two copies of $N$ such that all edges on the left side of $N$ incident to $N$ are now incident to one copy of $N$ and all edges on the right side of $N$ are incident with the other copy of $N$. We say that $G'$ is obtained from $G$ by cutting along $N$. The copies $N_X$ and $N_Y$ of $N$ are called **cut-nooses**.

**Proposition 1.** Let $G$ be a torus-embedded graph, let $G'$ be the graph obtained from $G$ by cutting along a noncontractible tight noose $N$ on $G$. Then $G'$ is planar. (cf. Figure 1)

**Definition 2. HAMILTONIAN-CYCLE-Problem:** Given a graph $G$, is there a cycle that visits each vertex exactly once?

**HAMILTOR:** HAMILTONIAN-CYCLE-Problem for a torus-embedded graph $G$.

Since a cut as shown in Figure 1 is along vertices, a possible Hamiltonian cycle in $G$ will be disconnected.

**Definition 3. A cut of a Hamiltonian cycle** $C$ in $G$ along a tight noose $N$ is a set of disjoint paths in $G'$.

After identifying the components of a Hamiltonian cycle in the planar graph $G'$ they can be used to reconstruct the Hamiltonian cycle in $G$. Let $N_X$ and $N_Y$ be the cut-nooses obtained from cutting $G$ along $N$. Also let $x_i \in N_X$ and $y_i \in N_Y$ be duplicated vertices of the same vertex in $N$.

**Definition 4 (Relaxed Hamiltonian Set of Paths).** We call a set of disjoint paths $P$ in $G'$ relaxed Hamiltonian if:

1. Every path has its endpoints in $N_X$ and $N_Y$.
2. Vertex $x_i$ is an endpoint of some path $P_i$ iff $y_i$ is an endpoint of a path $P' \neq P$.

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1 Figure 1 from Frederic Dorn, 'Designing Subexponential Algorithms: Problems, Techniques and Structures', Thesis for the degree of Philosophiae Doctor (PhD), Department of Informatics, UNIVERSITY OF BERGEN, Bergen, Norway July 2007, p.18
3. If one of $x_i, y_i$ is an inner vertex of a path, the other one is not in any path.

4. Every vertex of $G' \setminus N_X \cup N_Y$ is in some path.

A cut of a Hamiltonian cycle in $G$ must have a corresponding relaxed Hamiltonian set of paths in $G'$, but not vice versa. Checking in linear time for a single relaxed Hamiltonian set if it is also a solution for HAMILTOR is possible, by identifying the corresponding vertices of $N_X$ and $N_Y$. From this follows that checking all possible relaxed Hamiltonian sets in $G'$ will answer the question, whether a Hamiltonian cycle exists in $G$ or not, and will return one, in case of existence.

But only equivalence classes on the set $HS(G')$ of all of relaxed Hamiltonian sets have to be checked:

**Definition 5.** For any two sets $P_1, P_2 \in HS(G')$, $P_1 \sim P_2$ if for every path in $P_i$ there is a path in $P_j$ with the same endpoints ($1 \leq i, j \leq 2$).

**Lemma 1.** The number of different equivalence classes of relaxed Hamiltonian sets in $G'$ is in $O(2^{3k})$ where $k$ is the length of the tight noose $N$.

The proof of Lemma 1 is based upon the fact that the number of non-crossing paths with its endpoints in one noose corresponds in a way to the Catalan numbers. The two cut-nooses here are melted into one cut-noose by looking at one path in the relaxed Hamiltonian set and cutting the sphere along this path, cf. Figure 2.

For reconstructing a possible Hamiltonian cycle, we start with a set of vertex tuples in $G'$ called a candidate $K = \{(s_1, t_1), ..., (s_k, t_k)\}$ with $s_i, t_i \in N_X \cup N_Y$, $i = 1, ..., k$ and a vertex set $I \subset N_X \cup N_Y$. If there exists a relaxed Hamiltonian set $P$ such that every $(s_i, t_i)$ marks the endpoints of a path and the vertices of $I$ are inner vertices of some paths, we can rebuild a Hamiltonian cycle in $G$ from $P$. By using dynamic programming on Sphere-Cut-Decompositions it is checked for every candidate if there is a spanning subgraph of $G'$ isomorphic to a Hamiltonian set $P \sim K$.

The framework above can - after some careful modifications - also be used for solving non-local problems on graphs of bounded genus. The main result is that in all cases the resulting algorithm has running time $2^{O(\sqrt{n})}$.

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2Figure 2 from Frederic Dorn, Fedor V. Fomin, Dimitrios M. Thilikos: 'Fast subexponential algorithm for non-local problems on graphs of bounded genus', p.6
Parameterized problems in graphs embedded in surfaces of bounded genus can also be solved with this technique with running time $2^{O(\sqrt{p})} \cdot n^{O(1)}$ (for example: finding a cycle of length $p$, if it exists).