On the Number of Spanning Trees a Planar Graph can Have

Charilaos Zisopoulos

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Remember that a spanning tree of a connected undirected graph \( G = (V, E) \) is a minimal set of edges, that connects all vertices. Equivalently, it is a maximal set of edges, that contains no cycle.

Let \( Q \) be the Laplacian matrix of \( G \). Kirchhoff’s matrix tree theorem says that the number of spanning trees of \( G \) is equal to the absolute value of any cofactor of \( Q \).

We are interested in upper bounding the grid size needed to embed 3D-polytopes. To do this, it suffices to get upper bounds for the cofactors of the Laplacian. By Kirchhoff’s theorem, we can do this by upper bounding the number of spanning trees. In the figure below you can see an integer grid embedding of the dodecahedron.

Figure 1: Integer embedding of the dodecahedron

We define the edge graph \( G(P) \) of a convex polytope \( P \) as the connected graph whose vertex set is the vertex set of the polytope \( P \), and two vertices are adjacent in the graph if they are endpoints of a 1-face of \( P \).
Also, remember that a graph \( G = (V, E) \) is 3-connected if the graph remains connected when at most 2 vertices are deleted from the graph.

According to Steinitz’s theorem, a graph \( G \) is the edge graph of a convex 3D-polytope iff \( G \) is simple, planar and 3-connected. So, it suffices to obtain an upper bound on the number of spanning trees of planar and 3-connected graphs.

We define a face cycle in a graph \( G \) as a cycle with edges in the boundary of a face of \( G \), i.e. it cannot surround ”interior” vertices and edges. If this face cycle has exactly three edges, it is called a triangle. Note that not all 3-cycles are triangles, as demonstrated in the figure below.

![Figure 2: A 3-cycle that is not a triangle](image)

Let \( \Gamma_n \) denote the set of planar graphs with \( n \) vertices. Also let \( t(G) \) be the number of spanning trees of graph \( G \). Then the maximum number of spanning trees a graph \( G \in \Gamma_n \) can have is \( T(n) = \max_{G \in \Gamma_n} \{t(G)\} \). Our goal is to find some \( \alpha \) such that \( T(n) \leq \alpha^n \).

A triangulation is a maximal planar graph; if any edge was added the graph would no longer be planar. All the faces of such a graph are bounded by three edges, and if it has at least 4 vertices, it is also 3-connected. Therefore, the graph that will have \( t(G) = T(N) \) will be a triangulation. The idea of the paper presented is to exploit the graph’s planarity to provide an improved upper bound on \( T(n) \). This is achieved using the refined outgoing edge approach.

We call a directed graph outdegree-one if some vertex \( w \) has no outgoing arcs, and every other vertex is incident to exactly one outgoing arc.

By transforming each edge of \( G \) to a pair of directed arcs, we have that every spanning tree of \( G \) can be transformed into exactly one outdegree-one graph, by selecting edges and orienting them towards a node \( w \). Such an example is provided in figure 3.

The converse is not always true, since a outdegree-one graph can contain cycles. However, outdegree-one graphs without cycles are exactly the oriented spanning trees of \( G \). Thus, if \( P_{nc} \) is the probability that the random outdegree-one graph generated by picking an outgoing edge for each vertex of \( G \) uniformly contains no cycle, we have that:

\[
t(G) = \left( \prod_{i=2}^{n} d_i \right) P_{nc}
\]  \hspace{1cm} (1)
The next step is to express $P_{nc}$ in a way that allows to upper bound it. We do this by enumerating the cycles of $G$ and defining $C_i$ as the event that the $i$-th cycle occurs in a random outdegree graph and $C_i^c$ that it does not. Then if $s$ is the number of cycles in $G$, we can write:

$$P_{nc} = Pr[\cap_{j=1}^{s} C_j^c]$$  \hspace{1cm} (2)

The main lemma of the paper is that for all $i$, the events $C_i$ have mutually exclusive dependencies and union-closed independencies, which allows us to upper bound $P_{nc}$ by an expression that involves only products of the probabilities of events $C_i$ and $C_i^c$. Furthermore, we can only consider 2-cycles and triangles and still obtain an upper bound on $P_{nc}$, since considering more cycle lengths would make $P_{nc}$ smaller.

In order to make this approach feasible, we transform the problem into a linear program. To do this, we first construct signatures of the 2-extensions of 2- and 3-cycles. We are able to do this thanks to a theorem by Whitney, according to which if a graph is planar and 3-connected its facial structure is uniquely determined. An example is provided in figure 4 below.

We can use those signatures to describe whether two cycles are dependent. For each signature, we define an appropriate variable that will be used in the LP-program. Using several technical tricks, we are able to transform the expression that upper bounds $P_{nc}$ into a LP-program that uses these variables. The same variables are used to provide constraints for the LP-program, which always hold for planar graphs, but do not necessarily hold for non-planar graphs.

An important detail is that we use the same variables to introduce the expression $\prod_{i=2}^{s} d_i$ into the LP. We do this by using a charging scheme. The charging scheme works by distributing $di$ to the neighbors of $i$. We show an example of a charging scheme in figure 5 below.

The above approach gives a solution for a specific $n$, however we want to have a solution that is asymptotic for growing $n$. In order to do this we normalize the LP that we obtained and then we solve its dual, since the original LP would have infinitely many variables. This dual program on the other hand will have infinitely many constraints. Through a careful construction and by using the
Figure 4: Signatures of 2-extensions

Convention for naming the signatures of 2-extensions of 2-cycles (on the left) and 3-cycles (on the right).

Figure 5: A charging scheme

Every triangle incident to $v_i$ gets $\log j/i$ from $v_j$.

Every triangle incident to $v_j$ gets $\log j/i$ from $v_i$.

And every triangle incident to $v_k$ gets $\log k/k$ from $v_k$.

$$\sum_{i,j,k,A,B,C} f_{ij}(A,B,C) \left( \frac{\log i}{i} + \frac{\log j}{j} + \frac{\log k}{k} \right)$$
weak duality theorem, we are able to circumvent this and obtain the desired upper bound.

By applying the above approach, we get the following theorem:

**Theorem 1** Let $G$ be a planar graph with $n$ vertices. The number of spanning trees of $G$ is at most $O(5.28515^n)$.

If $G$ is 3-connected and contains no triangle, then the number of spanning trees is bounded by $O(3.41619^n)$.

If $G$ is 3-connected and contains no triangle and no quadrilateral, the number of its spanning trees is bounded by $O(2.71565^n)$.

As a corollary, we get the following result for embedding 3D-polytopes in integer grids:

**Corollary 1** The grid size needed to realize a 3D-polytope with integer coordinates is bounded by $O(147.7^n)$.

For grid embeddings of simplicial 3D-polytopes the bound is $O(27.94^n)$. 