Kuratowski’s Theorem and Whitney’s Duality

Giulio Malavolta

Graph on Surfaces
INTRODUCTION – PLANAR GRAPHS

A graph $G$ is **EMBEDDED** in a topological space $X$ if the vertices of $G$ are distinct elements of the space and every edge of $G$ is a simple arc connecting in $X$ the two vertices which it joins in $G$, such that its interior is disjoint from other edges or vertices

A graph is **PLANAR** if it can be drawn in the plane in such a way that no edges intersect
INTRODUCTION – POLYGONAL ARCS

An arc in a plane $R^2$ is a POLYGONAL ARC if it is the union of a finite number of straight segments.

If $G$ is a planar graph, then $G$ has a representation in the plane such that all edges are simple polygonal arcs.

Graph embedded in a plane
Define a space $\varepsilon$ around each vertex such that it includes only incident edges
Define $C$ as the straight line connecting each space $\varepsilon$; it's possible to draw all edges as polygonal arcs.
INTRODUCTION – CLOSED ARCS

If C is a CLOSED POLYGONAL ARC in the plane then it divides the space in two connected regions, also called FACES, having C as its boundary.

\[\pi(z) = \text{number of boundary segments touched by the horizontal half line at the point } z\]

\[\pi(z) \mod 2 = 1 \text{ inner vertex}\]
\[\pi(z) \mod 2 = 0 \text{ external vertex}\]

At LEAST two connected components!

At MOST two connected components!

Any three points can be connected by the boundary with polygonal arcs.
INTRODUCTION – EULER’S FORMULA

If G is a planar graphs where all edges are polygonal arcs, than G has exactly $V - E + F = 2$

Each face has a cycle of G as its boundary
INTRODUCTION – EULER’S APPLICATION

TRIANGULATION: connected plane graph with polygonal edges such that each face is a 3-CYCLE
QUADRANGULATION: connected plane graph with polygonal edges such that each face is a 4-CYCLE

In a triangulation every edge bounds 2 faces and every face is composed by 3 edges
$3f \leq 2e$

In a quadrangulation every edge bounds 2 faces and every face is composed by 4 edges
$4f \leq 2e$

$f = e - v + 2 \quad E \leq 3V - 6$
$f = e - v + 2 \quad E \leq 2V - 4$
INTRODUCTION – $K_5$ AND $K_{3,3}$

The complete graph $K_5$ and the bipartite graph $K_{3,3}$ are NON PLANAR

$5 \leq 30 - 6$ does NOT hold! $6 \leq 27 - 6$ does NOT hold!
Kuratowski’s Theorem
Lemma: “A graph is planar if and only if it does not contain a subdivision of $K_5$ or a subdivision of $K_{3,3}$ as a subgraph.”

A **SUBDIVISION** of a graph $G$ is a graph resulting from an addition of a new vertex between two vertices and the replacement of the edge with two new edges.
Kuratowski’s Theorem – Convex Embedding

A graph $G$ is **Straight Line Embedded** if each edge is a straight line segment.

If each bounded face of a straight embedded graph is convex then the embedding of $G$ is said to be **Convex**.

If $G$ is a **3-connected** graph with no subdivision of $K_{3,3}$ and $K_5$ as a subgraph, then $G$ has a **Convex Embedding** in the plane.
Kuratowski’s Theorem – Convex Embedding

G is a 3-connected graph with a concave face
Kuratowski’s Theorem – Convex Embedding

In a 3-connected graph there’s at least one edge that can be contracted keeping the graph 3-connected.
Deleting the edges incidents to $z$ we define the cycle $C$. 
Kuratowski’s Theorem – Convex Embedding

The neighbors of x define three paths along the cycle C
If the neighbors of $y$ only belong to a single path than the graph $G$ has a convex embedding.
KURATOWSKI’S THEOREM – CONVEX EMBEDDING

Convex embedding!
Kuratowski’s Theorem – $K_5$

**Subdivision**

What if the neighbors of $y$ are in more than one path?
Kuratowski’s Theorem – $K_5$

Subdivision
Kuratowski’s Theorem – $K_{3,3}$ Subdivision

What if the neighbors of $y$ are alternate to the neighbors of $x$?
Kuratowski’s Theorem – $K_{3,3}$ Subdivision

Subdivision of $K_{3,3}$!
Kuratowski’s Theorem – Generalization

If a graph $G$ of order $\geq 4$ contains no subdivision of $K_5$ or $K_{3,3}$ and the addition of any out of every the possible edges makes the graph non planar, then $G$ is 3-CONNECTED.

Kuratowski’s theorem is valid for 3-CONNECTED graphs.

Kuratowski’s theorem is valid for ALL graphs.
Whitney’s Duality
**WHITNEY’S DUALITY – 2-CONNECTED COMPONENTS**

A 2-CONNECTED GRAPH is a “non separable” graph such that no vertex is a cut, if any vertex were to be removed the graph remains connected.

A 2-CONNECTED COMPONENT is a 2-connected subgraph in a multigraph.

Each color corresponds to a 2-connected component.
**WHITNEY’S DUALITY – MINIMAL CUTS**

In a connected multigraph $G$ a set of edges $E$ is called **SEPARATING** if $G - E$ is disconnected in two non empty disjoint sets of vertices

A **CUT** (or separating edge set) $E$ is **MINIMAL** if no proper subset of $E$ is a separating set

[Diagram of a graph with minimal cut highlighted]
**WHITNEY’S DUALITY – MINIMAL CUTS**

Two edges $e_1$ and $e_2$ in a connected multigraph $G$ belong to a minimal cut if and only if $e_1$ and $e_2$ are in the same 2-connected component of $G$. 

Componet 1

Componet 2

Minimal cut!

Componet 1

Componet 2

Componet 3

NO Minimal cut!
**WHITNEY’S DUALITY – COMBINATORIAL DUAL**

Being $G$ a connected multigraph, $G^*$ is the **COMBINATORIAL DUAL** of $G$ if:

1) There is **ONE-BY-ONE CORRESPONDANCE** of edges

2) For each set of edges $E$ defining a **CYCLE** the combinatorial dual $E^*$ is a cut in $G^*$
WHITNEY’S DUALITY – GEOMETRIC DUAL

The geometric dual of the graph $G$ is defined as a graph $G^*$ with one \textbf{VERTEX} in each \textbf{FACE} of $G$ and an \textbf{EDGE} $E^*$ crossing each \textbf{EDGE} $E$ and joining the two vertices of the correspondent faces bounded by $e$
Whitney's Duality – geometric dual properties

If $E$ is an edge set defining a **cycle** in $G$, then the corresponding $E^*$ is a cut in $G^*$

G and geometric dual $G^*$

$G - E$ and geometric dual $G^*/E^*$
**Whitney’s Duality – Geometric Dual Properties**

If $E$ is the edge set of a **forest** in $G$, then $G^* - E^*$ is connected (no cut)

![Diagram](attachment:image.png)
Lemma: “Let $G$ be a 2-connected multigraph, then $G$ is planar if and only if it has a combinatorial dual. If $G^*$ is a combinatorial dual of $G$, then $G$ has an embedding in the plane such that $G^*$ is isomorphic to the geometric dual of $G$. In particular, also $G^*$ is planar, and $G$ is a combinatorial dual of $G^*$.”
Let $G$ be a simple cycle dividing the plane in two faces, any two edges $E^*$ are in a two cycle, therefore $G^*$ has only two vertices in its embedding in the plane.
We can represent any non cyclic graph with a cycle $G'$ and a path $P$ connecting two elements of $G'$.
Since P is not a cycle the correspondent set of edges $E^*$ cannot be a minimal cut, so the all the edges of the $G^*$ must be parallel joining two points $z_1$ and $z_2$.
The edges incident to the vertex $z_0$ represent a cut in $G^*$, therefore $G'$ is a cycle separating the vertex $z_0$ from $G^*-z_0$. 
We draw $P$ inside $G'$ defining two cycles $C_1$ and $C_2$ respectively containing subset of edges $E_1$ and $E_2$, that correspond to $E_1^*$ and $E_2^*$ defining a cut for $z_1$ and $z_2$. 
WHITNEY’S DUALITY – FROM COMBINATORIAL TO PLANARITY

The combinatorial dual implies an embedding in the plane of G

G is planar!
THANK YOU FOR THE ATTENTION!