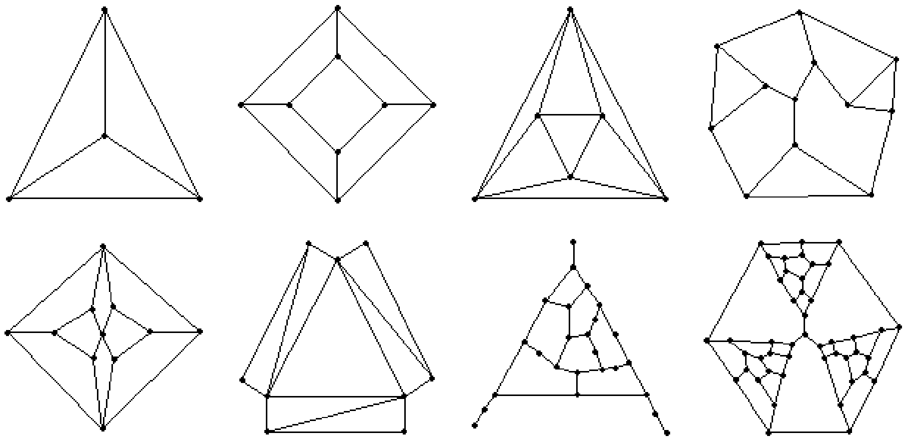
Kuratowski's Theorem and Whitney's Duality

Giulio Malavolta *Graph on Surfaces*

INTRODUCTION – PLANAR GRAPHS

A graph G is **EMBEDDED** in a topological space X if the vertices of G are distinct elements of the space and every edge of G is a simple arc connecting in X the two vertices which it joins in G, such that it's interior is disjoint from other edges or vertices

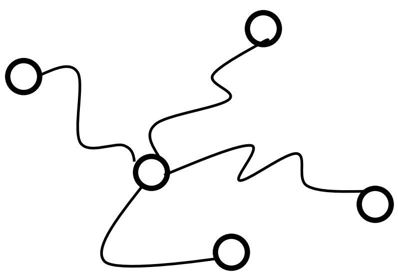


A graph is **PLANAR** if it can be drawn in the plane in such a way that no edges intersect

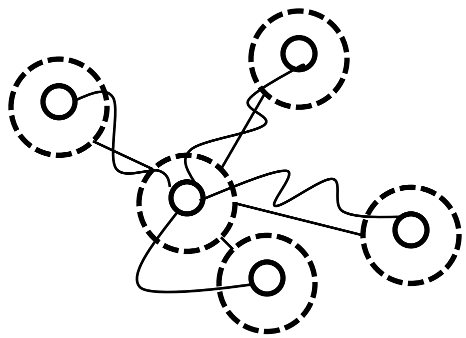
INTRODUCTION – POLYGONAL ARCS

An arc in a plane R^2 is a **POLYGONAL ARC** if is the union of a finite number of straight segments

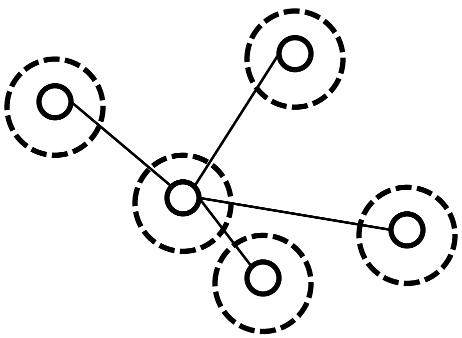
If G is a planar graph, than G has a representation in the plane such that all edges are simple polygonal arcs



Graph embedded in a plane



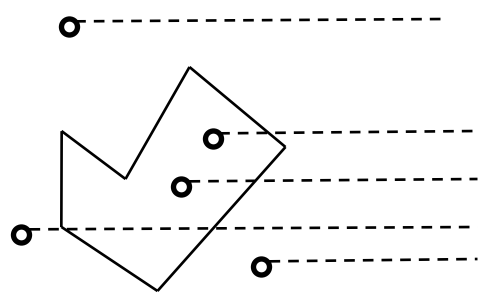
Define a space ε around each vertex such that it includes only incident edges



Define C as the straigh line connecting each space ε it's possible to draw all edges as polygonal arcs

INTRODUCTION – CLOSED ARCS

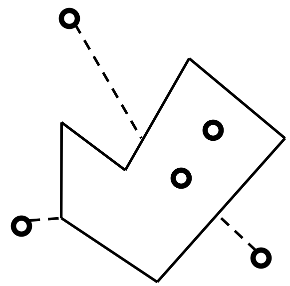
If C is a **CLOSED POLYGONAL ARC** in the plane then it divides the space in two connected regions, also called **FACES**, having C as its boundary



$\pi(z)$ = number of boundary segments touched by the the horizontal half line at the point z

$\pi(z) \bmod 2 = 1$ inner vertex
$\pi(z) \bmod 2 = 0$ external vertex

At **LEAST** two connected components!

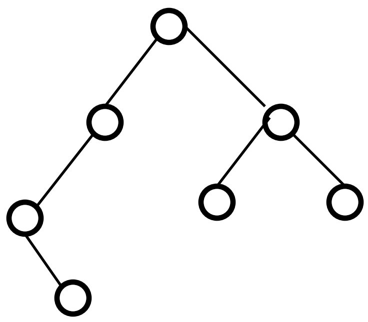


Any three points can be connected by the boundary with polygonal arcs

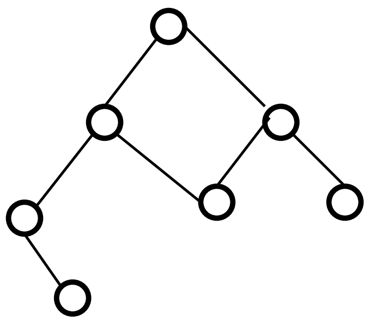
At **MOST** two connected components!

INTRODUCTION – EULER'S FORMULA

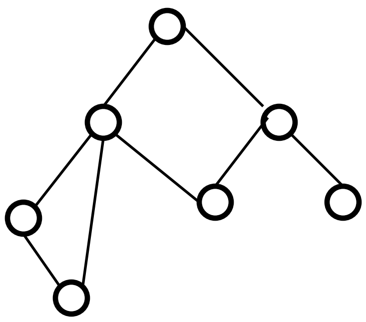
If G is a planar graphs where all edges are polygonal arcs, than G has exactly **VERTICES – EDGES + FACES = 2**



Vertices = 7
Edges = 6
Faces = 1



Vertices = 7
Edges = 7
Faces = 2



Vertices = 7
Edges = 7
Faces = 3

$$\mathbf{V - E + F = 2}$$

Each face has a cycle of G as its boundary

INTRODUCTION – EULER'S APPLICATION

TRIANGULATION: connected plane graph with polygonal edges such that each face is a **3-CYCLE**

QUADRANGULATION: connected plane graph with polygonal edges such that each face is a **4-CYCLE**

In a triangulation every edge bounds 2 faces and every face is composed by 3 edges
$3f <= 2e$

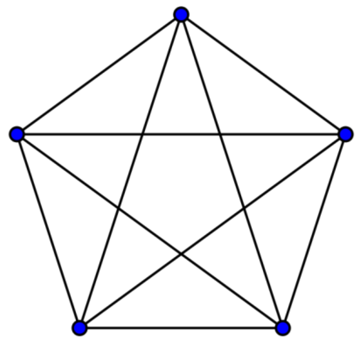
$\xrightarrow{f = e - v + 2}$ **E <= 3V - 6**

In a quadrangulation every edge bounds 2 faces and every face is composed by 4 edges
$4f <= 2e$

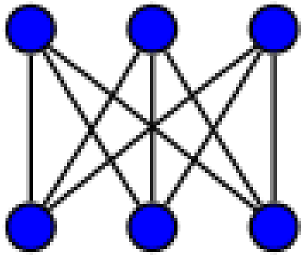
$\xrightarrow{f = e - v + 2}$ **E <= 2V - 4**

Introduction – K_5 and $K_{3,3}$

The complete graph $\mathbf{K_5}$ and the bipartite graph $\mathbf{K_{3,3}}$ are **NON PLANAR**



5 <= 30 – 6
does **NOT** hold!



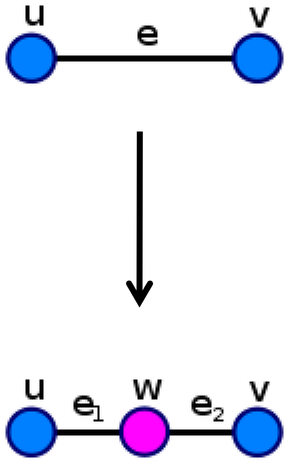
6 <= 27 – 6
does **NOT** hold!

Kuratowski's Theorem

KURATOWSKI'S THEOREM

Lemma: "*A graph is planar if and only if it does not contain a subdivision of K_5 or a subdivision of $K_{3,3}$ as a subgraph.*"

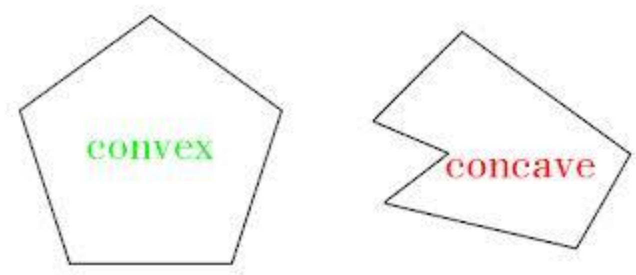
A **SUBDIVISION** of a graph G is a graph resulting from an addition of a new vertex between two vertices and the replacement of the edge with two new edges



Kuratowski's Theorem – convex embedding

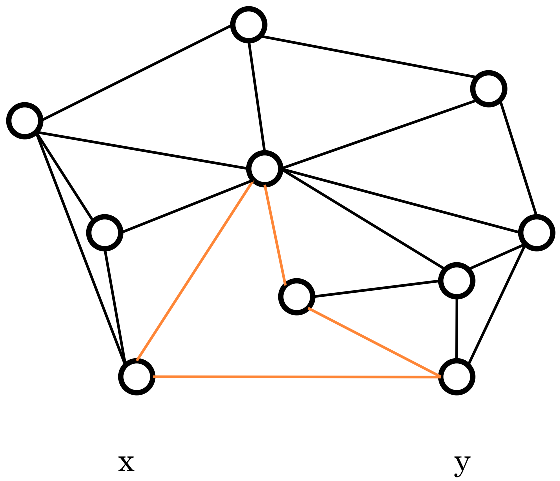
A graph G is **STRAIGHT LINE EMBEDDED** if each edge is a straight line segment

If each bounded face of a straight embedded graph is convex then the embedding of G is said to be **CONVEX**



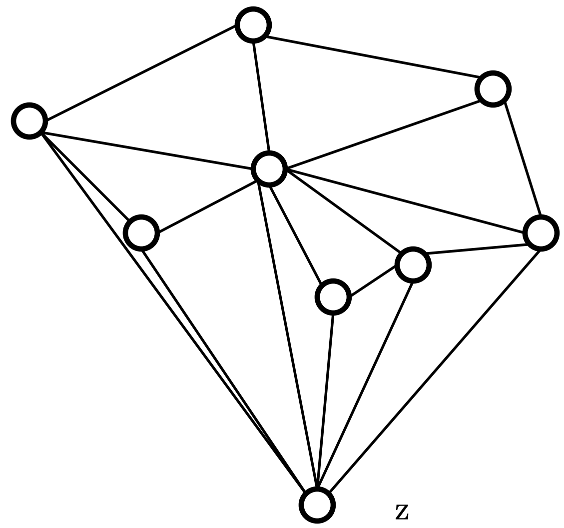
If G is a **3-CONNECTED** graph with no subdivision of K3,3 and K5 as a subgraph, then G has a **CONVEX EMBEDDING** in the plane

KURATOWSKI'S THEOREM – CONVEX EMBEDDING



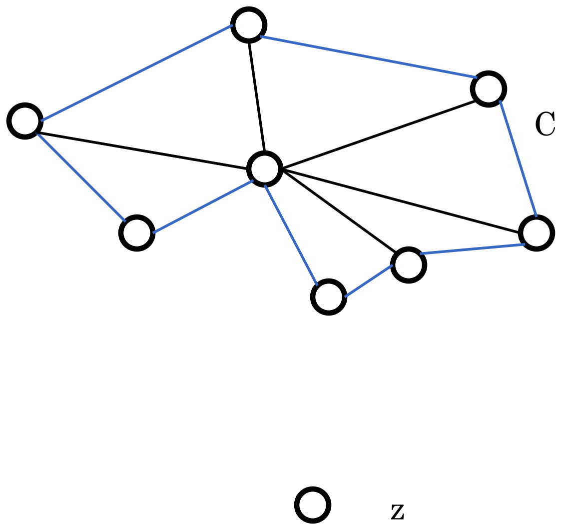
G is a 3-connected graph with a concave face

KURATOWSKI'S THEOREM – CONVEX EMBEDDING



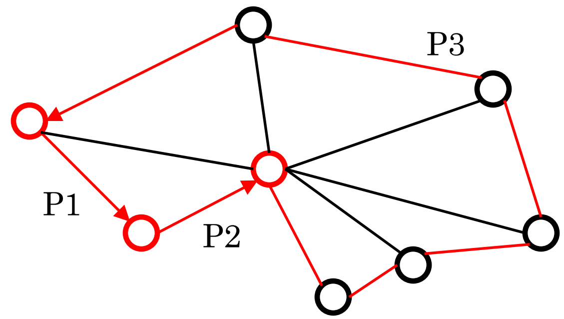
In a 3-connected graph there's at least one edge that can be contracted keeping the graph 3-connected

Kuratowski's Theorem – convex embedding



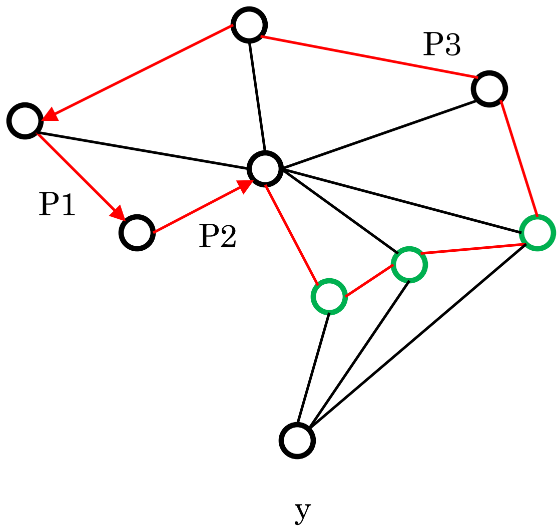
Deleting the edges incidents to z we define the cycle C

Kuratowski's Theorem – convex embedding



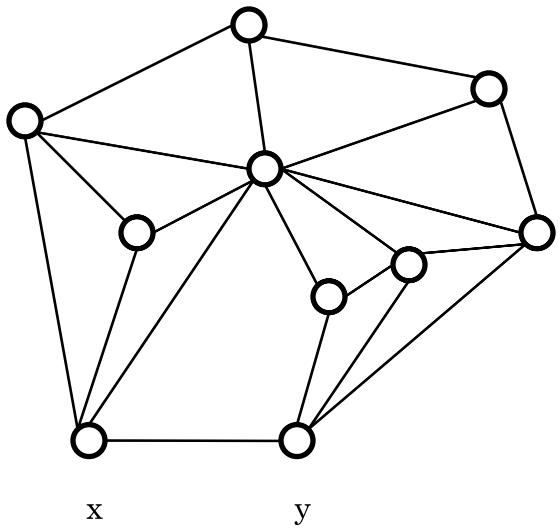
The neighbors of x define three paths along the cycle C

KURATOWSKI'S THEOREM – CONVEX EMBEDDING



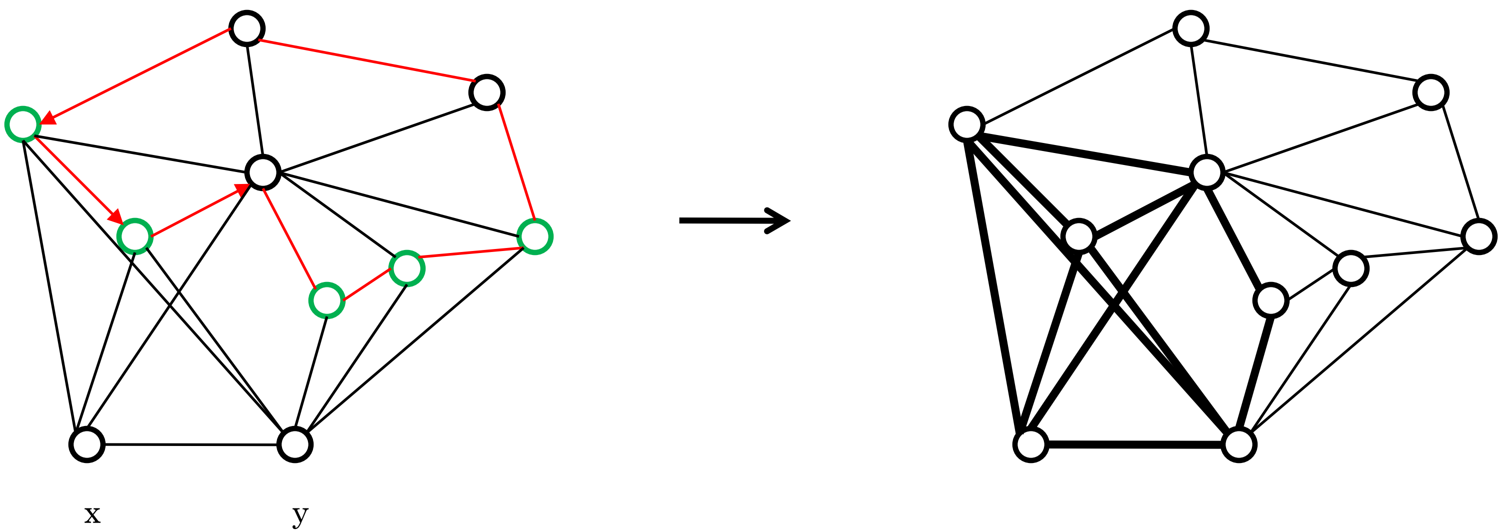
If the neighbors of y only belong to a single path than the graph G has a convex embedding

Kuratowski's Theorem – convex embedding



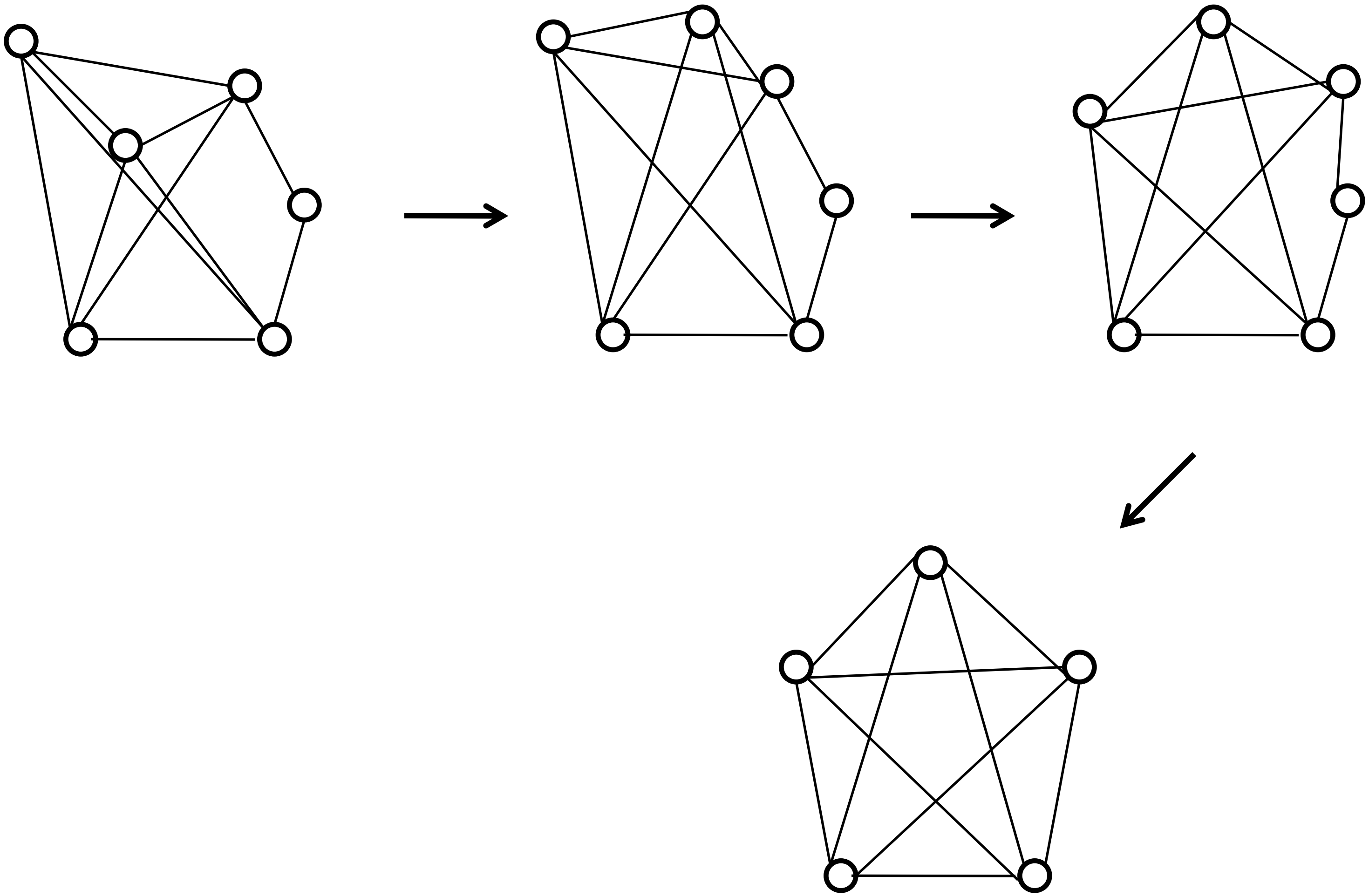
Convex embedding!

KURATOWSKI'S THEOREM – K_5 SUBDIVISION



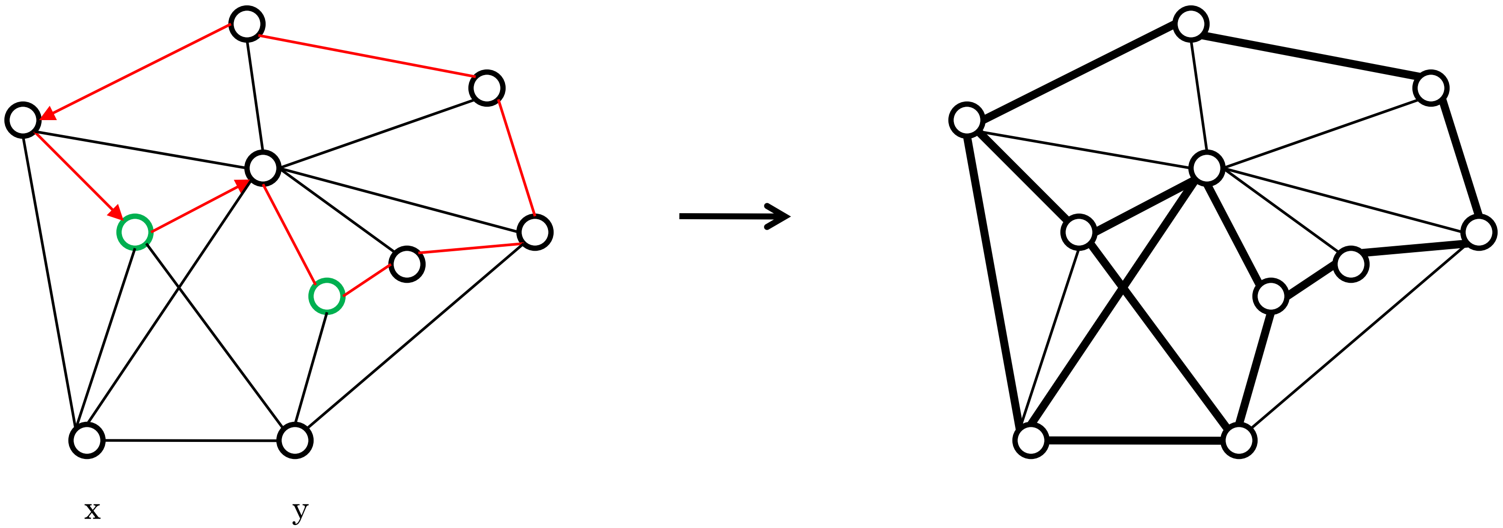
What if the neighbors of y are in more than one path?

Kuratowski's Theorem – K_5 subdivision



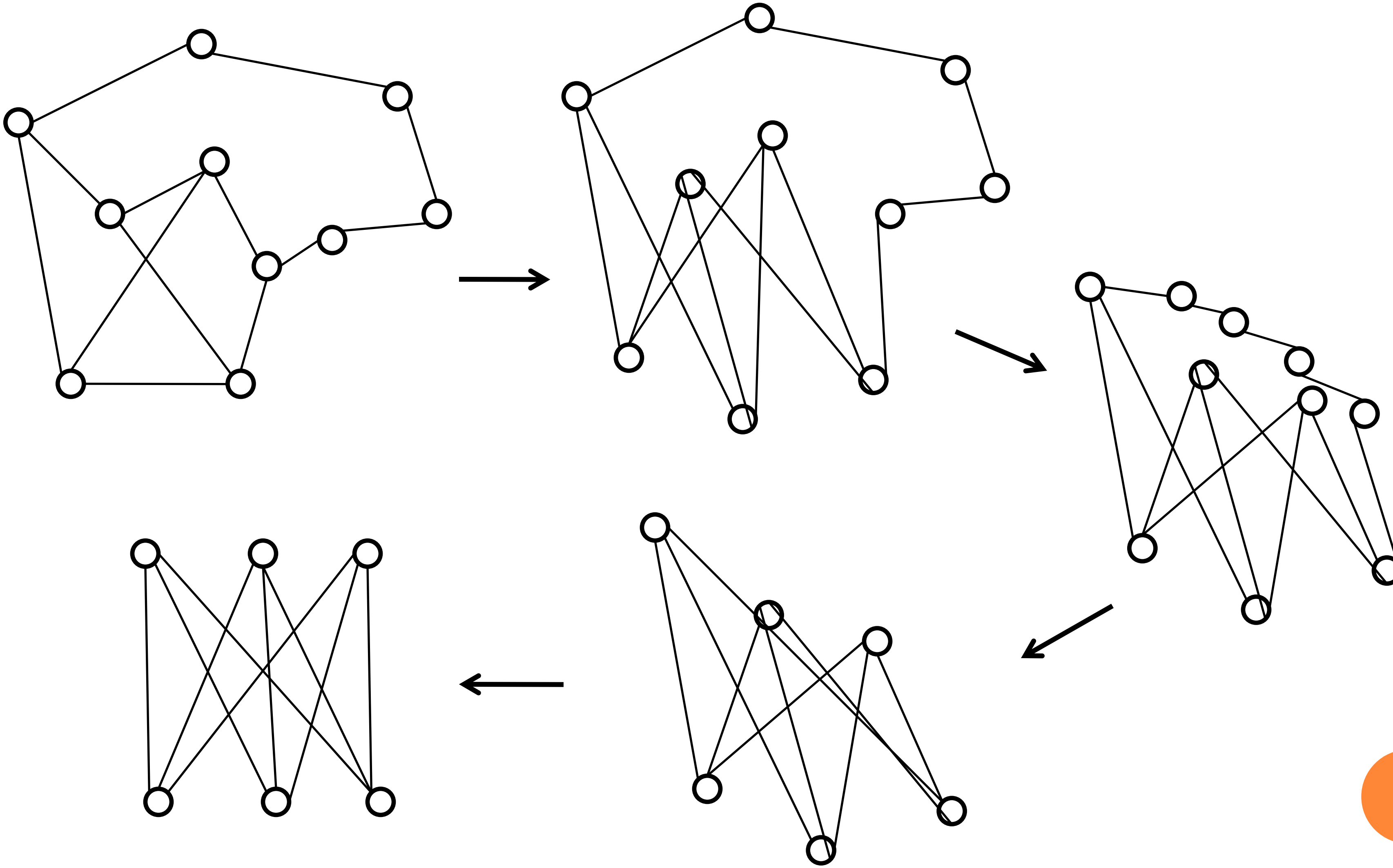
Subdivision of K_5!

KURATOWSKI'S THEOREM – $K_{3,3}$ SUBDIVISION



What if the neighbors of y are alternate to the neighbors of x?

Kuratowski's Theorem – $K_{3,3}$ subdivision



Subdivision of $K_{3,3}$!

KURATOWSKI'S THEOREM – GENERALIZATION

If a graph G of order >=4 contains no subdivision of K5 or K3,3 and the addition of any out of every the possible edges makes the graph non planar, then G is **3-CONNECTED**

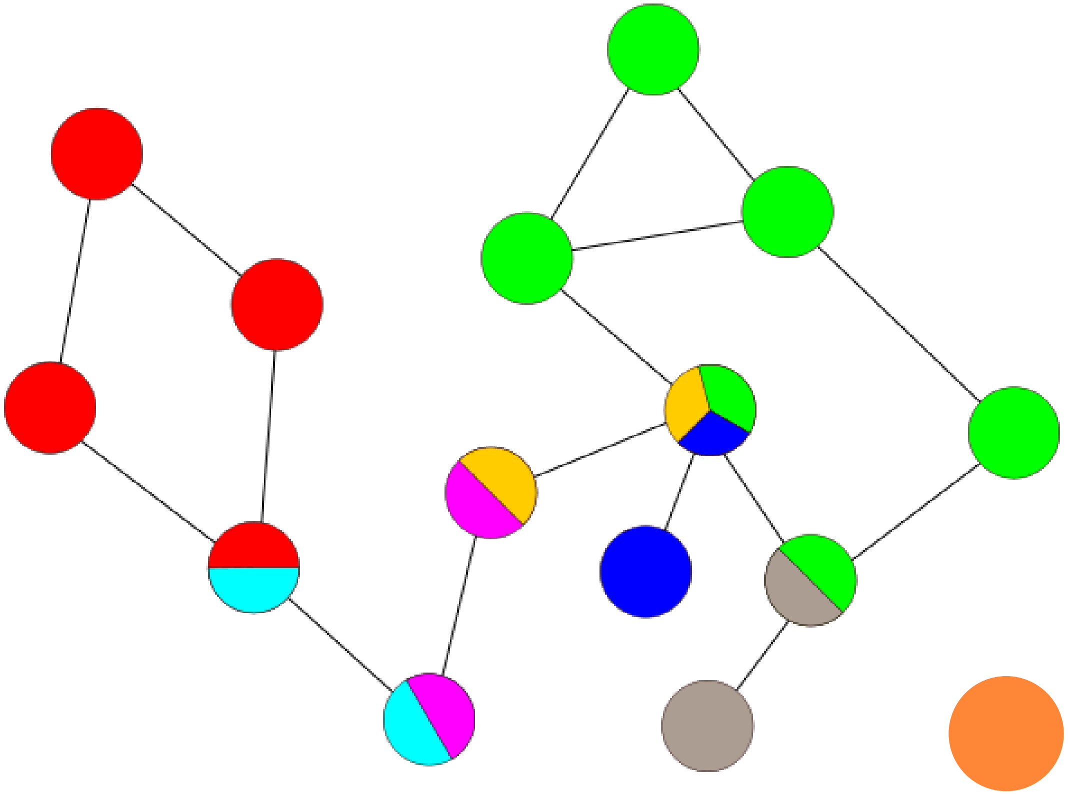
Kuratowsi's theorem is valid for **3-CONNECTED** graphs $\longrightarrow$ Kuratowsi's theorem is valid for **ALL** graphs

Whitney's Duality

WHITNEY'S DUALITY – 2-CONNECTED COMPONENTS

A **2-CONNECTED GRAPH** is a "non separable" graph such that no vertex is a cut, if any vertex were to be removed the graph remains connected

A **2-CONNECTED COMPONENT** is a 2-connected subgraph in a multigraph

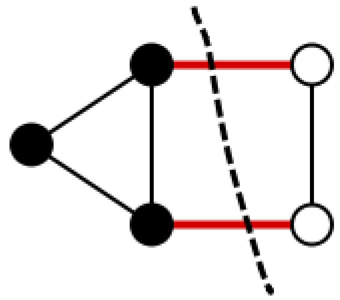


Each color corresponds to a 2-connected component

WHITNEY'S DUALITY – MINIMAL CUTS

In a connected multigraph G a set of edges E is called **SEPARATING** if G – E is disconnected in two non empty disjoint sets of vertices

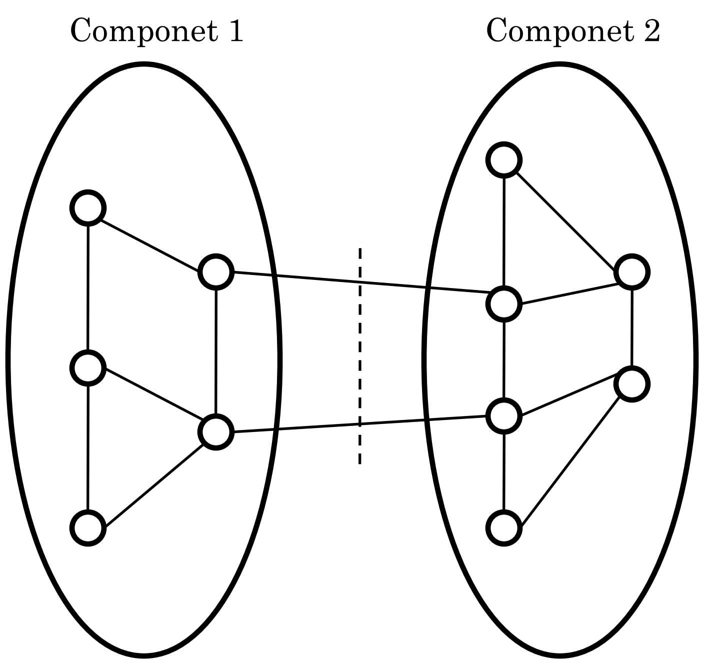
A **CUT** (or separating edge set) E is **MINIMAL** if no proper subset of E is a separating set



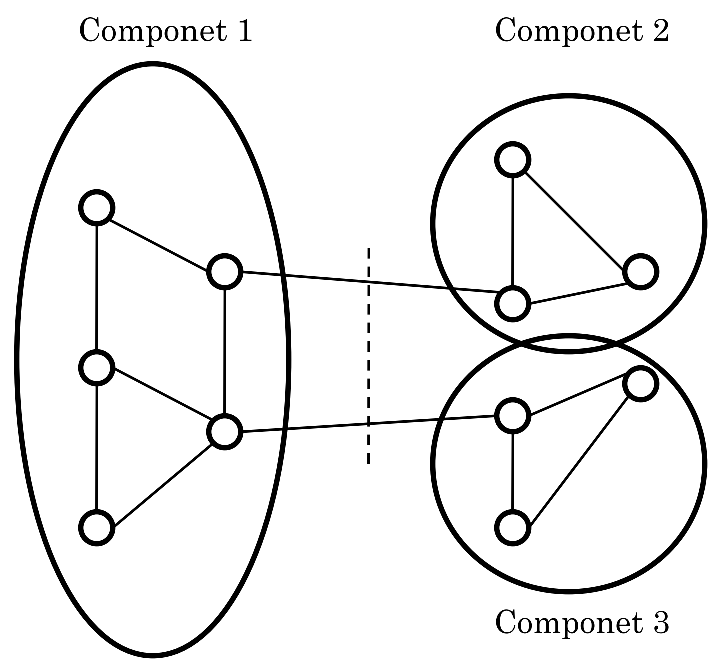
Minimal cut

Whitney's Duality – minimal cuts

Two edges e_1 and e_2 in a connected multigraph G belong to a minimal cut if and only if e_1 and e_2 are in the same 2-connected component of G



Minimal cut!



NO Minimal cut!

WHITNEY'S DUALITY – COMBINATORIAL DUAL

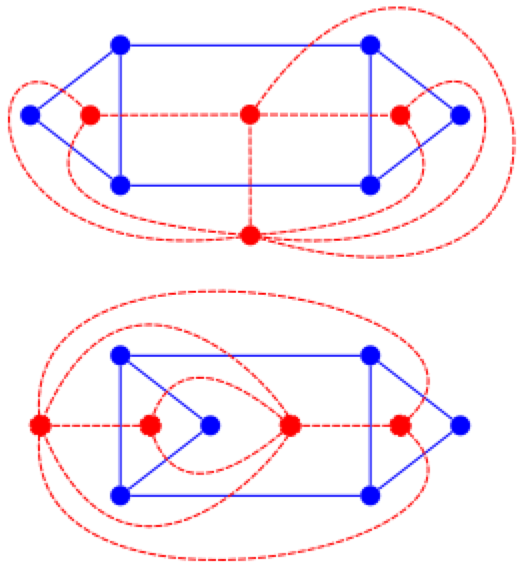
Being G a connected multigraph, G* is the **COMBINATORIAL DUAL** of G if:

1) There is **ONE-BY-ONE CORRESPONDANCE** of edges

2) For each set of edges E defining a **CYCLE** the combinatorial dual E* is a cut in G*

WHITNEY'S DUALITY – GEOMETRIC DUAL

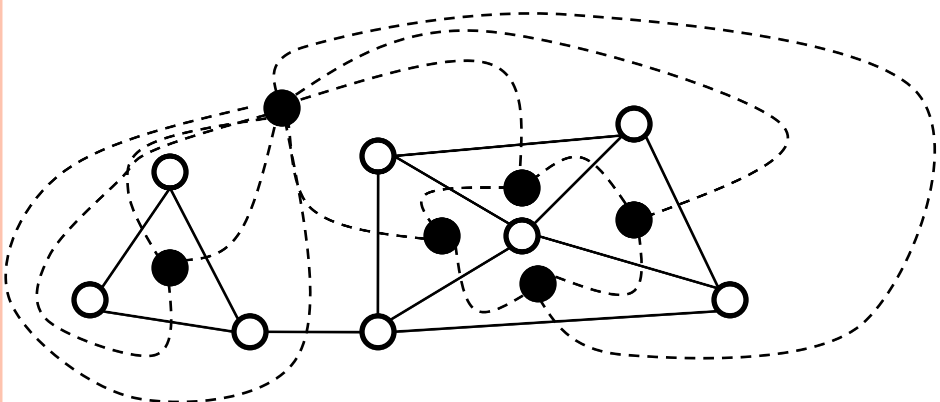
The geometric dual of the graph G is defined as a graph G* with one **VERTEX** in each **FACE** of G and an **EDGE E*** crossing each **EDGE E** and joining the two vertices of the correspondent faces bounded by e



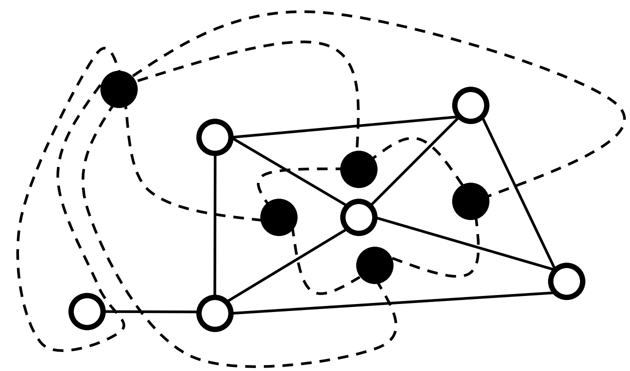
Geometric duals

Whitney's Duality – geometric dual properties

If E is an edge set defining a **CYCLE** in G, than the corresponding E* is a cut in G*



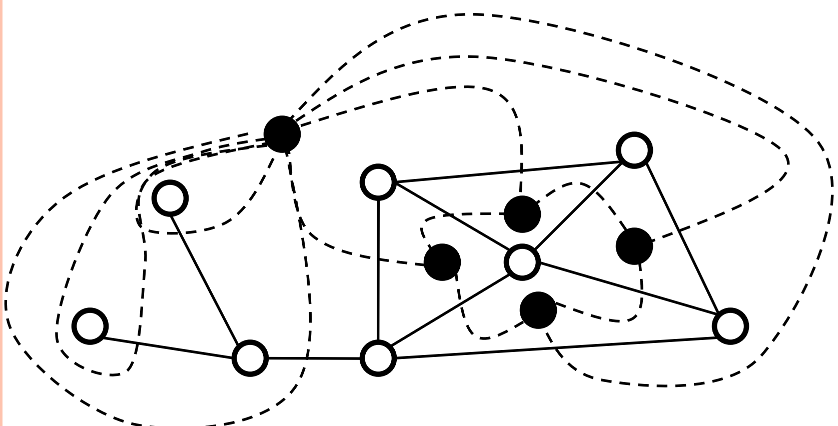
G and geometric dual G*



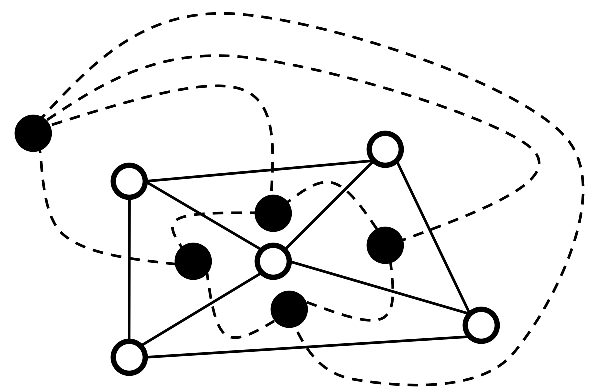
G – E and geometric dual G*/E*

WHITNEY'S DUALITY – GEOMETRIC DUAL PROPERTIES

If E is the edge set of a **FOREST** in G, then G* – E* is connected (no cut)



G and geometric dual G*



G – E and geometric dual G*/E*

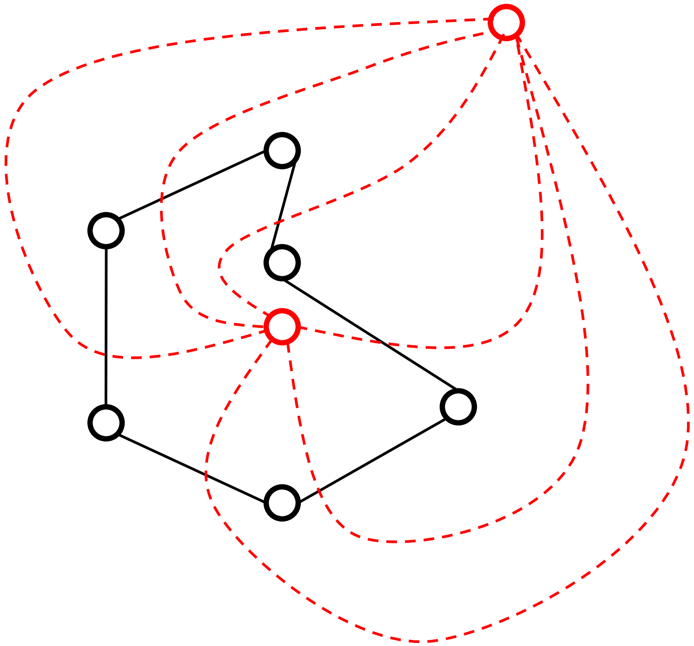
WHITNEY'S DUALITY

Lemma: *"Let G be a 2-connected multigraps, then G is planar if and only if it has a combinatorial dual. If G* is a combinatorial dual of G, than G has an embedding in the plane such that G* is isomorphic to the geometric dual of G. In particular, also G* is planar, and G is a combinatorial dual of G*."*

COMBINATORIAL DUAL ⟶ **GEOMETRIC DUAL** ⟶ **PLANARITY**

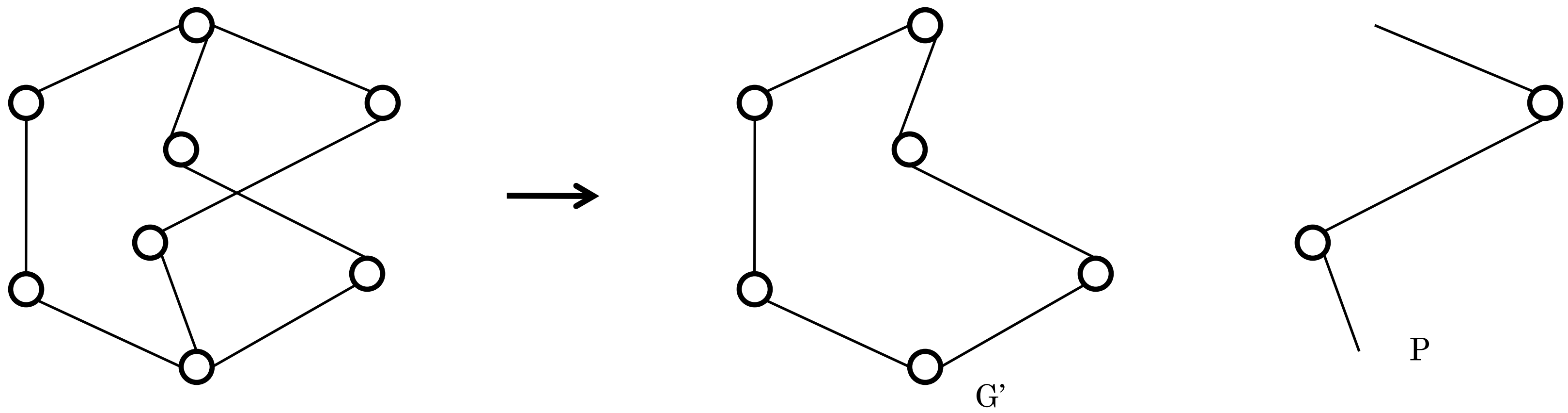
COMBINATORIAL DUAL ⟶ **PLANARITY**

Whitney's Duality – from combinatorial to planarity



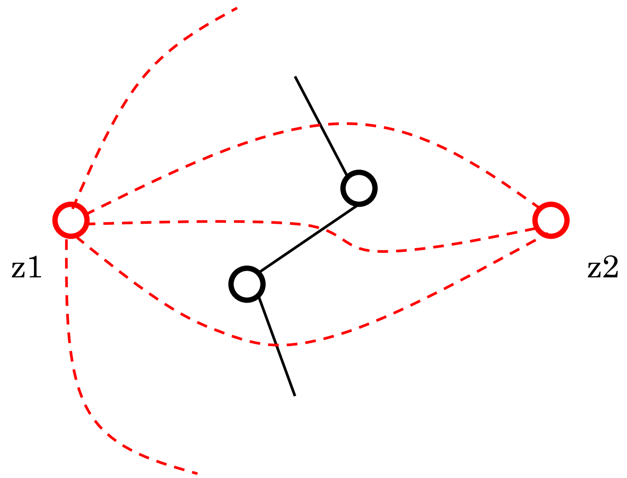
Let G be a simple cycle dividing the plane in two faces, any two edges E* are in a two cycle, therefore G* has only two vertices in its embedding in the plane

WHITNEY'S DUALITY – FROM COMBINATORIAL TO PLANARITY



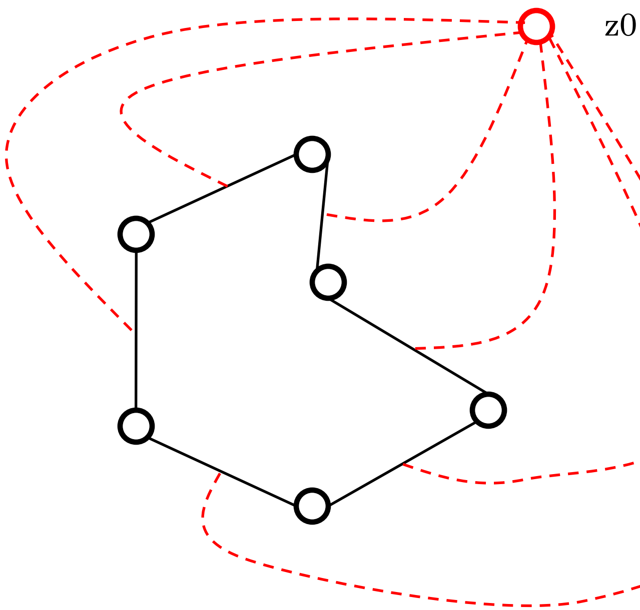
We can represent any non cyclic graph with a cycle G' and a path P connecting two elements of G'

Whitney's Duality – from combinatorial to planarity



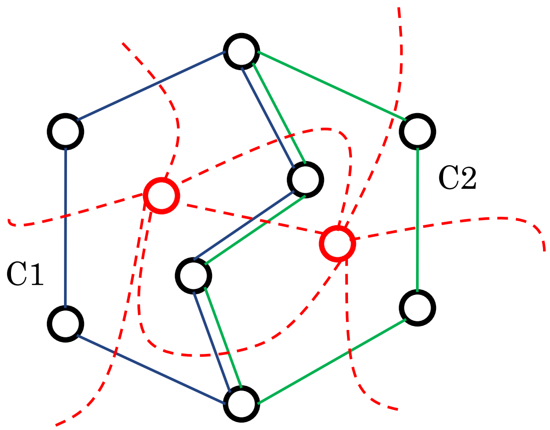
Since P is not a cycle the correspondent set of edges E* cannot be a minimal cut, so the all the edges of the G* must be parallel joining two points z1 and z2

Whitney's Duality – from combinatorial to planarity



The edges incident to the vertex z0 represent a cut in G*, therefore G' is a cycle separating the vertex z0 from G*-z0

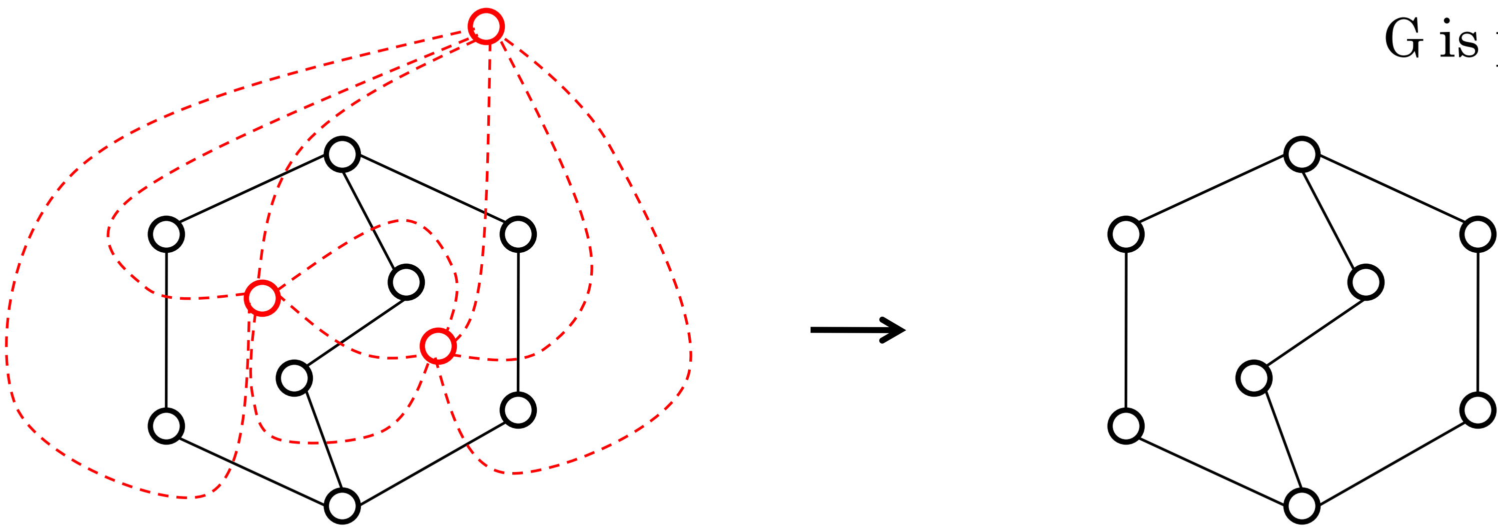
WHITNEY'S DUALITY – FROM COMBINATORIAL TO PLANARITY



We draw P inside G' defining two cycles C1 and C2 respectively containing subset of edges E1 and E2, that correspond to E1* and E2* defining a cut for z1 and z2

Whitney's Duality – from combinatorial to planarity

G is planar!



The combinatorial dual implies an embedding in the plane of G

Thank you for the attention!