A Combinatorial Polynomial Algorithm for the Linear Arrow-Debreu Market

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ABSTRACT

We present the first combinatorial polynomial time algorithm for computing the equilibrium of the Arrow-Debreu market model with linear utilities. Our algorithm views the allocation of money as flows and iteratively improves the balanced flow as in [Devanur et al. 2008] for Fisher's model. We develop new methods to carefully deal with the flows and surpluses during price adjustments. In our algorithm, we need $O(n^6 \log(nU))$ maximum flow computations, where n is the number of persons and U is the maximum integer utility, and the length of the numbers is at most $O(n \log(nU))$ to guarantee an exact solution. Previously, [Jain 2007] has given a polynomial time algorithm for this problem, which is based on solving a convex program using the ellipsoid algorithm.

1. INTRODUCTION

We provide the first combinatorial polynomial algorithm for computing the model of economic markets formulated by Walras in 1874 [12]. In this model, every person has an initial distribution of some goods and a utility function of all goods. The market clears at a set of prices if every person sells their initial goods and then uses their entire revenue to buy a bundle of goods with maximum utility. We want to find the market equilibrium in which every good is assigned a price so that the market clears. In 1954, two Nobel laureates, Arrow and Debreu [2], proved that the market equilibrium always exists if the utility functions are concave, which is why we call this model "Arrow-Debreu market". But, their proof is based on Kakutani's fixed point theorem and is nonconstructive. Since then, many algorithmic results studied the linear version of this model, that is, all utility functions are linear.

The first polynomial algorithm for the linear Arrow-Debreu model was found by Jain in 2007 [9]; it is based on solving a convex program using the ellipsoid algorithm and simultaneous diophantine approximation. Before that, Devanur and Vazirani [5] gave an approximation scheme for computing the Arrow-Debreu model with running time $O(\frac{n^4}{\epsilon} \log \frac{n}{\epsilon})$, improving [10]. Recently, Ghiyasvand and Orlin [8] improved the running time to $O(\frac{n}{\epsilon}(m + n \log n))$, where m is the number of pairs (i, j) such that buyer i has some utility for purchasing good j.

Many combinatorial algorithms consider a simpler model proposed by Fisher (see [3]). Eisenberg and Gale [6] reduced the problem of computing the Fisher market equilibrium to a concave cost maximization problem and thus gave the first polynomial algorithm for the Fisher market by ellipsoid algorithm. The first combinatorial polynomial algorithm for an exact linear Fisher market equilibrium is given by Devanur et al [4]. They use the maximum flow algorithm as a black box in their algorithm. When the input data is integral, their algorithm needs $O(n^5 \log U + n^4 \log e_{max})$ max-flow computations, where n is the number of buyers, U the largest integer utility, and e_{max} the largest initial amount of money of a buyer. If we use the common $O(n^3)$ max-flow algorithm (see [1]), their running time is $O(n^8 \log U + n^7 \log e_{max})$. Recently, Orlin [11] improved the running time for computing the linear Fisher model to $O(n^4 \log U + n^3 \log e_{max})$ and also gave the first strongly polynomial algorithm with running time $O(n^4 \log n)$.

Our results. In this paper, we extend the method in [4] to find a combinatorial algorithm for the linear Arrow-Debreu market. W.l.o.g., we can assume that each of the n persons has only one unit of goods, which is different from the goods other people have. When the price of that goods is p_i, the person will have a budget of p_i amount of money equivalently. It can be shown that each person only buys their favorite goods, that is, the goods with the maximum ratio of utility and price. We construct a graph on the nodes of buyers and goods as well as equality edges representing each buyer's favorite goods, and then find a maximum flow in it. In the case of market clearing prices, all the buyers and goods will be saturated. There will otherwise be surplus, so we need to increase the prices of some goods in order to decrease the surplus.

As in [4], we use the notion of "balanced flow", which is a maximum flow that balances the surplus of buyers. It is helpful for finding a cut in the residual graph. When we increase the prices of some goods, the buyers in possession of these goods will obtain more budget. Unlike [11], where the flows stay unchanged during price adjustment, we also increase the flows to those goods by the same ratio. So, some buyers will have more surplus, and some will have less. We can show that in some cases, the new flow is more balanced, that is, the l_2-norm of the surplus vector will decrease by a factor of $1 - \Omega(1/n^3)$. In other cases, some prices will increase by a factor of $1 + O(1/n^3)$. Since we can bound the number of the latter type of price adjustment iterations by a polynomial of n and $\log U$, the total running time is also polynomial.

2. MODEL AND DEFINIITONS

We make the following assumptions on the model as in Jain's paper [9]:

1. There are n persons in the system. Each person i has only one good, which is different from the goods other people have. The good person i has is denoted by good i.
2. Each person has only one unit of good. So, if the price of good i is p_i, person i will obtain p_i units of money when selling its good.
3. Each person i has a linear utility function $\sum_j u_{ij} z_{ij}$, where z_{ij} is the amount of good j consumed by i.
4. Each u_{ij} is an integer between 0 and U.
5. For every i, there is a j such that $u_{ij} > 0$. (Everybody likes some goods.)
6. For every j, there is an i such that $u_{ij} > 0$. (Every good is liked by somebody.)
7. For every proper subset P of persons, there exist $i \in P$ and $j \notin P$ such that $u_{ij} > 0$.

All these assumptions, with the exception of the last, are without loss of generality. The last assumption implies that all the equilibrium prices are nonzero [9], and it is only useful for the next section. In Section 4, we will discuss more about the last assumption.

Let $p = (p_1, p_2, ..., p_n)$ denote the vector of prices of goods 1 to n, so they are also the budgets of persons 1 to n. In this paper, we denote the set of all buyers to be $B = \{b_1, b_2, ..., b_n\}$ and the set of all goods to be $C = \{c_1, c_2, ..., c_n\}$. So, if the price of goods c_i is p_i, buyer b_i will have p_i amount of money. For a subset B' of persons or a subset C' of goods, we also use $p(B')$ or $p(C')$ to denote the total prices of the goods the persons in B' own or the goods in C'. For a vector $v = (v_1, v_2, ..., v_k)$, let:

- $|v| = |v_1| + |v_2| + ... + |v_k|$ be the l_1-norm of v.
- $\|v\| = \sqrt{v_1^2 + v_2^2 + ... + v_k^2}$ be the l_2-norm of v.

Clearly, every person is happiest with goods j, which maximizes the ratio u_{ij}/p_j. Define its *bang per buck* to be $\alpha_i = \max_j\{u_{ij}/p_j\}$. The classical Arrow-Debreu [2] theorem says that there is a non-zero market clearing price vector.

3. THE ALGORITHM

Our algorithm resembles [4], which finds a balanced flow and increases the prices in the "active subgraph". But, in the Arrow-Debreu model, when we increase the prices of some good i, the budget of buyer i will also increase. So, we need to find a careful way to prevent the total surplus from increasing.

Construct a flow network $G = (\{s,t\} \cup B \cup C, E_G)$, where s is the source node and t is the sink node, then $B = \{b_1, ..., b_n\}$ denotes the set of buyers and $C = \{c_1, ..., c_n\}$ denotes the set of goods. E_G consists of:

- Edges from s to every node b_i in B with capacity p_i.
- Edges from every node c_i in C to t with capacity p_i.
- Edges from b_i to c_j with infinite capacity if $u_{ij}/p_j = \alpha_i$. Call these edges "equality edges".

So, our aim is to find a price vector p such that there is a flow in which all edges from s and to t are saturated, i.e., $(s, C \cup B \cup t)$ and $(s \cup C \cup B, t)$ are both minimum cuts. When this is satisfied, all goods are sold and all of the money earned by every person is spent.

In a flow f, define the surplus $r(b_i)$ of a buyer i to be the residual capacity of the edge (s, b_i), and define the surplus $r(c_j)$ of a good j to be the residual capacity of the edge (c_j, t). That is, $r(b_i) = p_i - \sum_j f_{ij}$, and $r(c_j) = p_j - \sum_i f_{ij}$, where f_{ij} is the amount of flow in the edge (b_i, c_j). Define the surplus vector of buyers to be $r(B) = (r(b_1), r(b_2), ..., r(b_n))$. Also, define the total surplus to be $|r(B)| = \sum_i r(b_i)$, which is also $\sum_j r(c_j)$ since the total capacity from s and to t are both equal to $\sum_i p_i$. For convenience, we denote the surplus vector of flow f' by $r'(B)$. In the network corresponding to market clearing prices, the total surplus of a maximum flow is zero.

3.1 Balanced flow

As in [4], we define the concept of balanced flow to be a maximum flow that balances the surpluses of buyers. (However, unlike in their paper, the surpluses of goods can be positive here, which are not supposed to be balanced, so the balanced flow is not necessarily unique.)

Definition 1. In the network G of current p, a *balanced flow* is a maximum flow that minimizes $\|r(B)\|$ over all choices of maximum flows.

For flows f and f' and their surplus vectors $r(B)$ and $r'(B)$, respectively, if $\|r(B)\| < \|r'(B)\|$, then we say f is *more balanced* than f'. The next lemma shows why it is called "balanced".

LEMMA 1. *[4] If $a \geq b_i \geq 0, i = 1, 2, ..., k$ and $\delta \geq \sum_{i=1}^{k} \delta_i$, where $\delta, \delta_i \geq 0$, $i = 1, 2, ..., k$, then:*

$$\|(a, b_1, b_2, ..., b_k)\|^2 \leq \|a + \delta, b_1 - \delta_1, b_2 - \delta_2, ..., b_k - \delta_k\|^2 - \delta^2.$$

PROOF.

$$
\begin{aligned}
& (a+\delta)^2 + \sum_{i=1}^{k}(b_i-\delta_i)^2 - a^2 - \sum_{i=1}^{k} b_i^2 \\
\geq\ & 2a\delta + \delta^2 - 2\sum_{i=1}^{k} b_i\delta_i \\
\geq\ & \delta^2 + 2a(\delta - \sum_{i=1}^{k}\delta_i) \geq \delta^2.
\end{aligned}
$$

□

LEMMA 2. *[4] In the network G for a price vector p, given a maximum flow f, a balanced flow f' can be computed by at most n max-flow computations.*

PROOF. In the residual graph G_f w.r.t. to f, let $S \subseteq B \cup C$ be the set of nodes reachable from s, and let $T = (B \cup C) \setminus S$ be the remaining nodes. Then, there are no edges from $S \cap B$ to $T \cap C$ in the equality graph, and there is no flow from $T \cap B$ to $S \cap C$. The buyers in $T \cap B$ and the goods in $S \cap C$ have no surplus w.r.t. f, and this holds true for every maximum flow. Let G' be the network spanned by $s \cup S \cup t$, and let f' be the balanced maximum flow in G'. The f' can be computed by n max-flow computations. (Corollary 8.8 in [4] is applicable since $(s \cup S, t)$ is a min-cut in G'.) Finally, f' together with the restriction of f to $s \cup T \cup t$ is a balanced flow in G. □

The surpluses of all goods in f' are the same as those in f since we only balance the surplus of buyers.

3.2 Price adjustment

As in [4, 11], we need to increase the prices of some goods to get more equality edges. For a subset of buyers B_1, define its neighborhood $\Gamma(B_1)$ in the current network to be:

$$\Gamma(B_1) = \{c_j \in C | \exists b_i \in B_1, s.t. (b_i, c_j) \in E_G\}.$$

Clearly, there is no edge in G from B_1 to $C \setminus \Gamma(B_1)$. In a balanced flow f, given a surplus bound $S > 0$, let $B(S)$ denote the subset of buyers with surplus at least S, that is, $B(S) = \{b_i \in B | r(b_i) \geq S\}$.

LEMMA 3. *In a balanced flow f, given a surplus bound S, there is no edge that carries flow from $B \setminus B(S)$ to $\Gamma(B(S))$.*

PROOF. Suppose there is such an edge (b_i, c_j) that carries flow such that $b_i \notin B(S)$ and $c_j \in \Gamma(B(S))$. Then, in the residual graph, there are directed edges (b_k, c_j) and (c_j, b_i) with nonzero capacities in which $b_k \in B(S)$. However, $r(b_k) \geq S > r(b_i)$, so we can augment along this path and get a more balanced flow, contradicting that f is already a balanced flow. □

From Lemma 3, we can increase the prices in $\Gamma(B(S))$ by the same factor x without inconsistency. There is no edge from $B(S)$ to $C \setminus \Gamma(B(S))$, and the edges from $B \setminus B(S)$ to $\Gamma(B(S))$ are not carrying flow, and hence, there will be no harm if they disappear from the equality graph. If there are edges (b_i, c_j) and (b_i, c_k) where $b_i \in B(S), c_j, c_k \in \Gamma(B(S))$, then $u_{ij}/p_j = u_{ik}/p_k$. Since the prices in $\Gamma(B(S))$ are multiplied by a common factor x, u_{ij}/p_j and u_{ik}/p_k remain equal after a price adjustment. However, the goods in $C \setminus \Gamma(B(S))$ will become more attractive, so there may be edges from $B(S)$ to $C \setminus \Gamma(B(S))$ entering the network, and the increase of prices needs to stop when this happens. Define such a factor to be $X(S)$, that is,

$$X(S) = \min\{\frac{u_{ij}}{p_j} \cdot \frac{p_k}{u_{ik}} | b_i \in B(S), (b_i, c_j) \in E_G, c_k \notin \Gamma(B(S))\}.$$

So, we need $O(n^2)$ multiplications/divisions to compute $X(S)$. When we increase the prices of the goods in $\Gamma(B(S))$ by a common factor $x \leq X(S)$, the equality edges in $B(S) \cup \Gamma(B(S))$ will remain in the network. We will also need the following theorem to prevent the total surplus from increasing.

THEOREM 1. *Given a balanced flow f in the current network G and a surplus bound S, we can multiply the prices of goods in $\Gamma(B(S))$ with a parameter $x > 1$. When $x \leq \min_i\{p_i/(p_i - r(b_i)) | b_i \in B(S), c_i \notin \Gamma(B(S))\}$ and $x \leq X(S)$, we obtain a flow f' in the new network G' of adjusted prices with the same value of total surplus by:*

$$f'_{ij} = \begin{cases} x \cdot f_{ij} & \text{if } c_j \in \Gamma(B(S)); \\ f_{ij} & \text{if } c_j \notin \Gamma(B(S)). \end{cases}$$

Then, the surplus of each good remains unchanged, and the surpluses of the buyers become:

$$r'(b_i) = \begin{cases} x \cdot r(b_i) & \text{if } b_i \in B(S), c_i \in \Gamma(B(S)); \\ (1-x)p_i + x \cdot r(b_i) & \text{if } b_i \in B(S), c_i \notin \Gamma(B(S)); \\ (x-1)p_i + r(b_i) & \text{if } b_i \notin B(S), c_i \in \Gamma(B(S)); \\ r(b_i) & \text{if } b_i \notin B(S), c_i \notin \Gamma(B(S)). \end{cases}$$

We call these kinds of buyers type 1 to type 4 buyers, respectively.

PROOF. Since the flows on all edges associated with goods in $\Gamma(B(S))$ are multiplied by x, the surplus of each good in $\Gamma(B(S))$ remains zero. Only the surplus of type 2 buyers decreases because the flows from a type 2 buyer b_i are multiplied by x, but its budget p_i is not changed. The flow after adjustment is $x(p_i - r(b_i))$. We need this to be at most p_i, so $x \leq p_i/(p_i - r(b_i))$ for all type 2 buyers b_i, and in f', the new surplus $r'(b_i) = (1-x)p_i + xr(b_i)$.

Since both money and flows are multiplied by x for a type 1 buyer, his surplus is also multiplied by x. For a type 3 buyer b_i, his flows are not changed, but his money is multiplied by x, so the new surplus is $xp_i - (p_i - r(b_i))$. □

After each price adjustment, in the new network, we will find a maximum flow by augmentation on the adjusted flow f' and then find a balanced flow by Lemma 2. This will guarantee that when the surplus of a good becomes zero, it will not change to non-zero anymore. Thus, the prices of the goods with non-zero surplus will not be adjusted.

Property 1. The prices of goods with non-zero surpluses remain unchanged in the algorithm.

3.3 Whole procedure

The whole algorithm is shown in Figure 1, where K is a constant we will set later. In this section, one iteration denotes the execution of one entire iteration inside the loop. We will discuss the rounding and termination conditions in Section 3.4.

In the first iteration, we constructed a balanced flow f in the network where all prices are equal to 1. In the equality graph, we have at least one edge incident to every buyer. The total surplus will be bounded by n, actually $n-1$ as at least one good will be sold completely. From Theorem 1, in the execution of the algorithm, the total surplus will never increase.

To ensure that the algorithm will terminate in a polynomial number of steps, we will require the following lemmas. From Property 1, the prices of goods with surplus stay one during the whole algorithm, so there is still a good with price one in the end. And, we need to bound the largest price:

LEMMA 4. *The prices of goods are at most $(nU)^{n-1}$.*

PROOF. It is enough to show that during the entire algorithm, for any non-empty and proper subset $\hat{C}$ of goods, there are goods $c_i \in \hat{C}, c_j \notin \hat{C}$ such that $p_i/p_j \le nU$. So, when we sort all the prices in decreasing order, the ratio of two adjacent prices is at most nU. Since there is always a good with price 1, the largest price is $\le (nU)^{n-1}$.

If $\hat{C}$ contains goods with surpluses, then their price is 1. The claim follows.

Let $\hat{B} = \Gamma(\hat{C})$ be the set of buyers adjacent to goods in $\hat{C}$ in the equality graph. If there exist b_i, c_j s.t. $b_i \in \hat{B}, c_j \notin \hat{C}$ and $u_{ij} > 0$, let $c_k \in \hat{C}$ be one of the goods adjacent to b_i in the equality graph, and then $u_{ij}/p_j \le u_{ik}/p_k$. So, $p_k/p_j \le u_{ik}/u_{ij} \le U$.

If there do not exist such b_i, c_j, then there is no flow between $\hat{B}$ and $C \setminus \hat{C}$, and there is $b_k \notin \hat{B}$, but $c_k \in \hat{C}$. Otherwise the persons whose goods are in $\hat{C}$ will not like any goods not in $\hat{C}$, contradicting assumption (7). Let $B' = \{j | b_j \in \hat{B}, c_j \notin \hat{C}\}$ and $B'' = \{j | b_j \notin \hat{B}, c_j \in \hat{C}\}$. Then, there is a $j \in B'$ with $p_j \ge p(B')/n$, and hence,

$$\begin{aligned} p_k &\le p(B'') = p(\hat{C}) - p(\{j | b_j \in \hat{B}, c_j \in \hat{C}\}) \\ &\le p(\hat{B}) - p(\{j | b_j \in \hat{B}, c_j \in \hat{C}\}) \\ &= p(B') \le np_j. \end{aligned}$$

The inequality of the second line holds since goods in $\hat{C}$ have surplus 0 and all of the flows from $\hat{B}$ go to $\hat{C}$. □

By Lemma 9, we can round to the exact solution when the algorithm terminates. To analyze the correctness and running time, we need the following lemma:

LEMMA 5. *After every price adjustment by x, the l_2-norm of the surplus vector $\|r(B)\|$ will either*

- *be multiplied by a factor of $1 + O(1/n^3)$ when $x = 1 + \frac{1}{Kn^3}$, or*
- *be divided by a factor of $1 + \Omega(1/n^3)$.*

Note that by Lemma 8, the rounding procedure can only increase $\|r(B)\|$ by a factor of $1+O(1/n^4)$ since $S \ge \epsilon/(e \cdot n)$. (We will leave the discussion of Lemma 8 later, so we can ignore it in the analysis.)

THEOREM 2. *In total, we need to compute $O(n^6 \log(nU))$ maximum flows, and the length of numbers is bounded by $O(n \log(nU))$. Thus, if we use the common $O(n^3)$ max-flow algorithm (see [1]), the total running time is $O(n^{10} \log^2(nU))$.*

PROOF. By Lemma 4, every price can be multiplied by $x = 1 + \frac{1}{Kn^3}$ for $O(\log_{1+1/Kn^3}(nU)^n) = O(n^4 \log(nU))$ times, so the total number of iterations of the first type is $O(n^5 \log(nU))$. The total factor multiplied to $\|r(B)\|$ by the first type iterations is $(1 + O(1/n^3))^{O(n^5 \log(nU))}$.

At the beginning, $\|r(B)\| \le \sqrt{n}$. When the algorithm terminates, $\|r(B)\| < \epsilon = \frac{1}{4n^4U^{3n}}$, so the number of second type iterations is bounded by

$$\begin{aligned} & \log_{1+\Omega(1/n^3)}\left(\frac{1}{\epsilon}\sqrt{n}(1 + O(1/n^3))^{O(n^5 \log(nU))}\right) \\ = \; & O(n^5 \log(nU)). \end{aligned}$$

Thus, the total number of iterations performed is bounded by $O(n^5 \log(nU))$. Since we need to compute n max-flows for the balanced flow in every iteration, we need $O(n^6 \log(nU))$ maximum flow computations in total. By Lemma 4 and Lemma 8, the prices are rational numbers $\le (nU)^{n-1}$ and with denominator $\le U^n \Delta = 4n^9 U^{4n}$. Thus, the length of the numbers to be handled is bounded by $O(n \log(nU))$. Note that max-flow computations only need additions and subtractions. We perform multiplications and divisions when we scale prices and when we set up the max-flow computation in the computation of balanced flow. The numbers of multiplications/divisons is by a factor n less than the numbers of additions/subtractions, and hence, it suffices to charge $O(n \log(nU))$ per arithmetic operation. □

Next we will prove Lemma 5. When we sort all the buyers by their surpluses $b_1, b_2, ..., b_n$, b_1 is at least $|r(B)|/n$ (where $|r(B)|$ is the total surplus). So, for the first i in which $\frac{r(b_i)}{r(b_{i+1})} > 1 + 1/n$, we can see $\frac{r(b_j)}{r(b_{j+1})} \le 1 + 1/n$ for $j < i$, so $r(b_i) \ge r(b_1)(1 + 1/n)^{-n} > |r(B)|/(e \cdot n)$. When such an i does not exist, each $r(b_i)$ is larger than $|r(B)|/(e \cdot n)$, and all goods in $\Gamma(B)$ must have zero surplus because the flow is otherwise not maximum. Thus, there are goods that have no buyers, and hence, either new equality edges emerge, or x reaches $1 + \frac{1}{Kn^3}$ (condition (3a) below).

From the algorithm, in every iteration, x satisfies the following conditions:

(1) $x \le 1 + \frac{1}{Kn^3}$.

```
Initially set p_i = 1 for all goods i;
Repeat
    Construct the network G for the current p, and compute the balanced flow f in it;
    Sort all buyers by their surpluses in decreasing order: b_1, b_2, ..., b_n;
    Find the first i in which r(b_i)/r(b_{i+1}) > 1 + 1/n, and i = n when there is no such i;
    Let the surplus bound S = r(b_i) and obtain B(S), Γ(B(S)), X(S); (B(S) = {b_1, b_2, ..., b_i})
    Multiply the prices in Γ(B(S)) by a gradually increasing factor x > 1 until:
    (Let f' be the flow corresponding to x which is constructed according to Theorem 1.)
        New equality edges emerge (x reaches X(S));
        OR the surplus of a buyer ∈ B(S) and a buyer ∉ B(S) equals in f';
        OR x reaches 1 + 1/(K·n^3)
    Round the prices in Γ(B(S)) according to Lemma 8 with Δ = 4n^9 U^{3n};
Until |r(B)| < ϵ, where ϵ = 1/(4n^4 U^{3n});
Finally, round the prices according to Lemma 9 to get an exact solution.
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Figure 1: The whole algorithm

(2) In f', $r'(b) \geq r'(b')$ for all $b \in B(S), b' \notin B(S)$. Here, $r'(b)$ is the surplus of b w.r.t. f', the flow corresponding to x by Theorem 1.

(3) If $x < 1 + \frac{1}{Kn^3}$, the following possibilities arise:

(a) There is a new equality edge (b_i, c_j) with $b_i \in B(S), c_j \notin \Gamma(B(S))$. By Lemma 6 below, we can obtain a flow f'' in which either $r''(b_i) = r'(b_i) - p_j$, or there is a $b_k \notin B(S)$ with $r''(b_i) = r''(b_k)$ (same as (b)).

(b) When x satisfies the second requirement in the algorithm, it satisfies: there exists $b \in B(S)$ and $b' \notin B(S)$ such that $r'(b) = r'(b')$ in f'.

Lemma 6. *If there is a new equality edge (b_i, c_j) with $b_i \in B(S), c_j \notin \Gamma(B(S))$, we can obtain a flow f'' from f' in which either $r''(b_i) = r'(b_i) - p_j$, or there is a $b_k \notin B(S)$ with $r''(b_i) = r''(b_k)$.*

Proof. Let $B' \subseteq B \setminus B(S)$ be the set of buyers with flows to c_j in f', and let w be the largest surplus of a buyer in $B \setminus B(S)$. Run the following procedure (f'' denotes the current flow in the algorithm):

```
Augment along (b_i, c_j) gradually until:
    r''(b_i) = w or r''(c_j) = 0;
If r''(b_i) = w then Exit;
For all b_k ∈ B' in any order
    Augment along (b_i, c_j, b_k) gradually until:
        r''(b_i) = max{r''(b_k), w} or f''(b_k, c_j) = 0;
    Set w = max{r''(b_k), w};
    If r''(b_i) = w then Exit.
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During the procedure, the surplus of b_i decreases but cannot become less than the surplus of a buyer in $B \setminus B(S)$, so condition (2) holds. In the end, if $r''(b_i) = w$, then there is a $b_k \in B \setminus B(S)$ s.t. $r''(b_i) = r''(b_k)$; otherwise, c_j has no surplus, and the flow to it all comes from b_i, so $r''(b_i) = r'(b_i) - p_j$. □

From Theorem 1, the surpluses in f' will increase for type 1 and 3 buyers, will decrease for type 2 buyers, and will stay unchanged for type 4 buyers. Note that the surplus of a type 1 or 2 buyer cannot be smaller than the surplus of any type 3 or 4 buyer. From Theorem 1 and condition (2), we infer that the total surplus will not increase, type 2 and 3 buyers will get more balanced, and $r'(b) = x \cdot r(b)$ for type 1 buyers b, so $\|r'(B)\| \leq x\|r(B)\| = (1 + O(1/n^3))\|r(B)\|$.

In (3a), there is a new equality edge (b_i, c_j). After the procedure described in Lemma 6, if there is no $b_k \notin B(S)$ such that $r''(b_i) = r''(b_k)$, then $r''(b_i) = r'(b_i) - p_j$ ($p_j \geq 1$). For all $b_k \notin B(S)$, $r''(b_i) > r''(b_k)$, and $r''(b_k) = r'(b_k) + \delta_k$, where $\delta_k \geq 0$ and $\sum_{b_k \notin B(S)} \delta_k \leq p_j$. Because $|r(B)| \leq n$, $\|r(B)\|^2 \leq n^2$. By Lemma 1,

$$\begin{aligned} \|r''(B)\|^2 &\leq \|r'(B)\|^2 - p_j^2 \\ &\leq x^2\|r(B)\|^2 - 1 \\ &\leq x^2\|r(B)\|^2 - \frac{1}{n^2}\|r(B)\|^2 \\ &= (1 - \Theta(1/n^2))\|r(B)\|^2. \end{aligned}$$

So, we have $\|r''(B)\| = (1 - \Omega(1/n^2))\|r(B)\|$.

In (3a), after the procedure described in (3a), if there exists $b_k \notin B(S)$ such that $r''(b_i) = r''(b_k)$, then we are in a similar situation as (3b), possibly with an even smaller total surplus. So, we can prove this case by the proof of (3b).

In (3b), let $u_1, u_2, ..., u_k$ and $v_1, v_2, ..., v_{k'}$ be the list of original surpluses of type 2 and 3 buyers, respectively. Define $u = \min\{u_i\}, v = \max\{v_j\}$, so $u_i \geq u$ for all i, and $v_j \leq v$ for all j, and $u > (1 + 1/n)v$. After the price and flow adjustments in Theorem 1, the list of surpluses will be $u_1 - \delta_1, u_2 - \delta_2, ..., u_k - \delta_k$ and $v_1 + \delta'_1, v_2 + \delta'_2, ..., v_k + \delta'_{k'}$ (here $\delta_i, \delta'_j \geq 0$ for all i, j), and there exist I, J such that $u_I - \delta_I = v_J + \delta'_J$, where $u_I - \delta_I$ is the smallest among $u_i - \delta_i$, and $v_J + \delta'_J$ is the largest among $v_j + \delta'_j$ by condition (2). Since the surpluses of type 1 edges also increase, we have

$\sum_i \delta_i \geq \sum_j \delta'_j$, $\delta_I \leq \sum_i \delta_i$, and $\delta'_J \leq \sum_j \delta'_j$. Compute:

$$\begin{aligned}
& \sum_i (u_i - \delta_i)^2 + \sum_j (v_j + \delta'_j)^2 - (\sum_i u_i^2 + \sum_j v_j^2) \\
= & -2\sum_i u_i\delta_i + 2\sum_j v_j\delta'_j + \sum_i \delta_i^2 + \sum_j {\delta'}_j^2 \\
\leq & -u\sum_i \delta_i + v\sum_j \delta'_j - \sum_i \delta_i(u_i - \delta_i) + \sum_j \delta'_j(v_j + \delta'_j) \\
\leq & -(u-v)\sum_i \delta_i - (u_I - \delta_I)\sum_i \delta_i + (v_J + \delta'_J)\sum_j \delta'_j \\
\leq & -(u-v)\sum_i \delta_i \\
\leq & -(u-v)\max\{\delta_I, \delta'_J\} \\
\leq & -\frac{1}{2}(u-v)^2 \\
< & -\frac{1}{2(n+1)^2}u^2.
\end{aligned}$$

Let $w_1, w_2, ...w_{k''}$ be the list of surpluses of type 1 buyers; all of them are $\leq e \cdot u$. After price adjustment, the surpluses will be $x \cdot w_1, x \cdot w_2, ...x \cdot w_{k''}$ from Theorem 1. Compute:

$$\begin{aligned}
& \sum_i (xw_i)^2 \\
\leq & (1 + \frac{1}{Kn^3})^2 \sum_i w_i^2 \\
\leq & \sum_i w_i^2 + (\frac{2}{Kn^3} + \frac{1}{K^2n^6}) \cdot ne^2u^2 \\
= & \sum_i w_i^2 + (\frac{2}{Kn^2} + \frac{1}{K^2n^5})e^2u^2.
\end{aligned}$$

Let $K = 32e^2$, then the change to the sum of squares of surpluses for type 2 and 3 buyers is less than $-\frac{1}{8n^2}u^2 = -\frac{4}{Kn^2}e^2u^2$. The total change to $\|r(B)\|^2$ is:

$$(-\frac{2}{Kn^2} + \frac{1}{K^2n^5})e^2u^2.$$

Since $u \geq \frac{1}{e}r(b_i)$ for all buyers b_i, $nu^2 \geq \frac{1}{e^2}\|r(B)\|^2$. Since the change is negative, we have:

$$\begin{aligned}
\|r'(B)\|^2 & \leq \|r(B)\|^2 + (-\frac{2}{Kn^2} + \frac{1}{K^2n^5})\frac{1}{n}\|r(B)\|^2 \\
& = \|r(B)\|^2 - \frac{2}{Kn^3}\|r(B)\|^2 + \frac{1}{K^2n^6}\|r(B)\|^2 \\
& = \|r(B)\|^2(1 - \frac{1}{Kn^3})^2.
\end{aligned}$$

Thus, Lemma 5 is proved.

3.4 Rounding and termination condition

In this section, we will show how to round the prices to rational numbers with denominators of length $O(n\log(nU))$. Also, we need the rounding process to obtain an exact market equilibrium when the surplus is very small. Here, we define the *equality graph* F on $B \cup C$ of *undirected* equality edges between buyers and goods, and we consider every connected component in this undirected equality graph.

LEMMA 7. *In a connected component Ψ containing k goods in the equality graph, if we know that p_j is a rational number with denominator N, where $c_j \in \Psi \cap C$, then all the prices of goods in $\Psi \cap C$ are rational numbers with denominator $\leq N \cdot U^k$.*

PROOF. Find a tree that connects all the goods in Ψ. The tree will contain $k + k' - 1$ edges if it contains k' buyers. Then, we can get $k - 1$ linear independent equations $p_j/u_{ij} = p_{j'}/u_{ij'}$ when both (b_i, c_j) and $(b_i, c_{j'})$ are tree edges. Together with the equation $p_j = I/N$ for some integer I, we can see that all the prices of goods in Ψ have denominator $\leq N \cdot U^k$. ☐

Given an integer Δ, we call a connected component in the equality graph *consistent* if it has a good whose price is a rational number with denominator Δ. Then, by Lemma 7, the prices of goods in a consistent connected component are rational numbers with denominator $\leq \Delta \cdot U^n$.

LEMMA 8. *In a balanced flow f in the network, given $\Delta > nU^n$ and a surplus bound $S \geq n^5/\Delta$, if all the connected components in $(B \setminus B(S), C \setminus \Gamma(B(S)))$ are consistent, we can adjust the prices in $\Gamma(B(S))$ so that all connected components in the equality graph are consistent. In the adjusted flow f' by Theorem 1, $\|r'(B)\| = \|r(B)\|(1+O(\frac{1}{S\cdot\Delta}))$, where $r'(B)$ is the surplus vector in f'.*

PROOF. The procedure is shown below:

Set $B' = B(S)$;
Repeat
  Multiply the prices in $\Gamma(B')$ by $x > 1$ until:
    A price in $\Gamma(B')$ has denominator Δ;
    OR a new equality edge emerges;
  Update the flow f by Theorem 1;
  Remove new consistent components from $B' \cup \Gamma(B')$;
Until $B' = \emptyset$.

Since all the prices change by at most $1/\Delta$, the changes to the total surplus of type 2 buyers is at most $n/\Delta < S$, so the surplus of a type 2 buyer is still positive. In addition, the surplus added to each type 1 or 3 buyer is at most $1/\Delta$, so

$$\begin{aligned}
& \|r'(B(S))\|^2 \\
\leq & \|r(B(S))\|^2 + \frac{2}{\Delta}|r(B(S))| + \frac{n}{\Delta^2} \\
\leq & \|r(B(S))\|^2 + \frac{2}{S\Delta}\|r(B(S))\|^2 + \frac{n}{S^2\Delta^2}\|r(B(S))\|^2 \\
= & \|r(B(S)\|(1 + O(\frac{1}{S\Delta})).
\end{aligned}$$

During the algorithm, we can see that all the connected components in $(B \setminus B', C \setminus \Gamma(B'))$ are consistent since we move the new consistent components to it. When we find new equality edges connecting B' and $C \setminus \Gamma(B')$, some nodes in $B' \cup \Gamma(B')$ will connect to $(B \setminus B', C \setminus \Gamma(B'))$, so these nodes can be removed. When a price in $\Gamma(B')$ has denominator

Δ, the component containing it will become consistent, so the loop will run for at most n times. In each loop, we need to compute $O(n^2)$ multiplications/divisions, so the running time for this rounding procedure is less than the computation of a balanced flow. $\square$

LEMMA 9. *When the total surplus is* $< \frac{1}{4n^4U^{3n}} = \epsilon$ *in a flow* f*, we can obtain an exact solution from the current equality graph.*

PROOF. Add the edge (b_i, c_i) for each person i to the equality graph F to obtain F'. For a connected component of F', the sum of the prices on both sides are the same. For every component Φ of F' with no surplus node, increase its prices by a common factor until a new equality edge emerges; this will unite two components. Repeat this until all components in F' have a surplus node. We may assume w.l.o.g. that F' becomes connected by this process. Otherwise, the following argument can be applied independently to each component of F'. The total surplus is still less than ϵ. The following rounding procedure will be performed on these revised prices.

Denote the set of connected components in the equality graph (after the adjustments of the previous paragraph) by $\Lambda = \{\Psi_k\}$. For each component Ψ_k in F, find a spanning tree T_k in it, then write the following equations:

$$p_j/u_{ij} = p_{j'}/u_{ij'}, \forall (b_i, c_j), (b_i, c_{j'}) \in T_k.$$

Since we have one such equation if c_j and $c_{j'}$ are connected by one b_i, we can have $|\Psi_k \cap C| - 1$ linear independent equations. The total number of linear independent equations for all components in F is $n - |\Lambda|$.

Since there is no flow between components, for each component Ψ_k in F, the money difference between buyers and goods in Ψ_k is only the surplus difference. So, we can write

$$\sum_{b_i \in B \cap \Psi_k} p_i - \sum_{c_i \in C \cap \Psi_k} p_i = \epsilon_k, \forall \Psi_k.$$

Here, ϵ_k (positive or negative) comes from the surpluses of goods and buyers, so $\sum |\epsilon_k| \leq 2\epsilon$. If b_i and c_i belong to distinct connected components Ψ_j and Ψ_k, the coefficient of p_i is $+1$ in the equation of Ψ_j, -1 in the equation for Ψ_k, and 0 in all other equations. If b_i and c_i belong to the same connected component, the coefficient of p_i is zero in all equations. Assume now that there is a proper subset of the equations that is linear dependent. Then, if b_i or c_i belongs to one of the components in the subset, both of them do. However, the subset of components is a proper subgraph of F', and hence, there is at least one i such that only one of b_i or c_i belongs to the subset of components. Thus, we have $|\Lambda| - 1$ independent equations.

Since there is a good c_i with non-zero surplus, we have $p_i = 1$. Thus, the current price vector p is the solution of these linear equations $Ap = X$ in which A is invertible.

Consider the following n linear equations with ϵ_k removed:

$$p'_j/u_{ij} = p'_{j'}/u_{ij'}, \quad \forall \quad (b_i, c_j), (b_i, c_{j'}) \in T_k$$
$$\sum_{b_i \in B \cap \Psi_k} p'_i - \sum_{c_i \in C \cap \Psi_k} p'_i = 0, \quad \forall \quad \Psi_k$$
$$p'_i = 1, \quad \exists \quad r(c_i) > 0.$$

They can be denoted by $Ap' = X'$, so there is also a unique solution. The solution will be rational numbers with a common denominator $D \leq nU^n$ by Cramer's rule. Since $||X| - |X'|| < 2\epsilon$, the difference $|p'_i - p_i|$ of solutions of each price is at most $2\epsilon \cdot nU^n = \frac{1}{2n^3U^{2n}}$ by Cramer's rule. The difference between any two numbers of denominators $D, D' \leq nU^n$ is a rational number of denominator $D \cdot D' < n^2U^{2n}$, which is larger than $2|p'_i - p_i|$. Since p'_i is a rational number with denominator $D \leq nU^n$, we can get p'_i by rounding p_i to the nearest rational number of denominator $\leq nU^n$. This can be done by continued fraction expansion, which needs $O(n \log^2 D) = O(n^3 \log^2(nU))$ time by Theorem 3.13 in [7]. We can also compute $D = det(A)$ directly and round every price to the nearest rational with denominator D or solve the linear equations $Ap' = X'$. By Theorem 5.12 in [7], computing the determinant of a matrix of dimension n with entries $\leq U$ takes $\tilde{O}(n^4 \log U)$ time, and solving $Ap' = X'$ also takes $\tilde{O}(n^4 \log U)$ time.

Now, all the prices p'_i are of the form q_i/D, where q_i, D are integers and $D \leq nU^n$ a common denominator. So, $|p_i - \frac{q_i}{D}| \leq \frac{1}{2n^3U^{2n}} = \frac{\epsilon'}{D}$, in which $\epsilon' = \frac{D}{2n^3U^{2n}} \leq \frac{1}{2n^2U^n}$. Construct the flow network G' for the new prices $q = (q_1, q_2, ..., q_n)$. Consider any $b_i \in B$ and $c_j, c_k \in C$ and assume $u_{ij}/p_j \leq u_{ik}/p_k$. Then,

$$\begin{aligned} u_{ij}q_k &\leq u_{ij}(p_kD + \epsilon') \\ &\leq u_{ik}p_jD + u_{ij}\epsilon' \\ &\leq u_{ik}(q_j + \epsilon') + u_{ij}\epsilon' \\ &\leq u_{ik}q_j + (u_{ik} + u_{ij})\epsilon' \\ &< u_{ik}q_j + 1, \end{aligned}$$

and hence, $u_{ij}q_k \leq u_{ik}q_j$ since $u_{ij}q_k$ and $u_{ik}q_j$ are integral. We conclude that the edges in G are all in G'.

Denote the size of the cuts $(s, B \cup C \cup t)$ and $(s \cup B \cup C, t)$ in G' by Z, which is an integer. Then, the size of this cut in G is $\geq (Z - n\epsilon')/D$. If there is another cut in G' of size $\leq Z - 1$, it is also a cut in G, and its size in G is $\leq \frac{Z-1}{D} + 2n\frac{\epsilon'}{D} = Z/D - \frac{1-2n\epsilon'}{D}$, so the maximum flow in G will have total surplus $> \frac{1-3\epsilon'}{D} > \epsilon$. Thus, $(s, B \cup C \cup t)$ and $(s \cup B \cup C, t)$ are both min-cuts in G', so the prices reach a market equilibrium. $\square$

4. GENERAL CASE

Here, we consider the case which does not satisfy the assumption (7) in Section 2. We draw the linking graph of persons in which there is a directed edge from i to j iff $u_{ij} > 0$. If the graph is strongly connected, then the case satisfies assumption (7). Otherwise, we can shrink each connected component into one vertex, then the graph will be a DAG, and we can find a topological order of strongly connected components: $P_1, P_2, ...P_k$, in which there are only edges from a lower order to a higher order. We use the algorithm in Section 3 to compute the equilibrium for all the persons in every strongly connected component P_i ($i = 1, 2, ..., k$). For $i = 2, ..., k$, multiplying the prices in P_i by $(U + 1) \cdot \max\{p_j | j \in P_{i-1}\}$ will ensure that there are no equality edges from P_i to P_j for $i < j$. Since the persons in P_j do not like any goods in P_i for $i < j$, this will not affect the equilibrium of every component, so we get a global equilibrium.

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