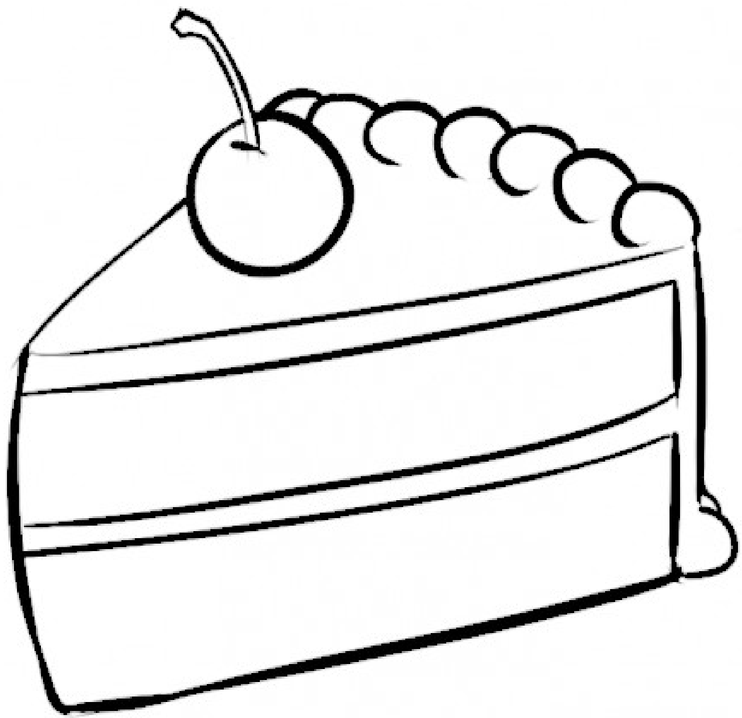
On Maxsum Fair Cake Divisions

Mark Simkin

19.12.2012

Motivation

Motivation



Outline

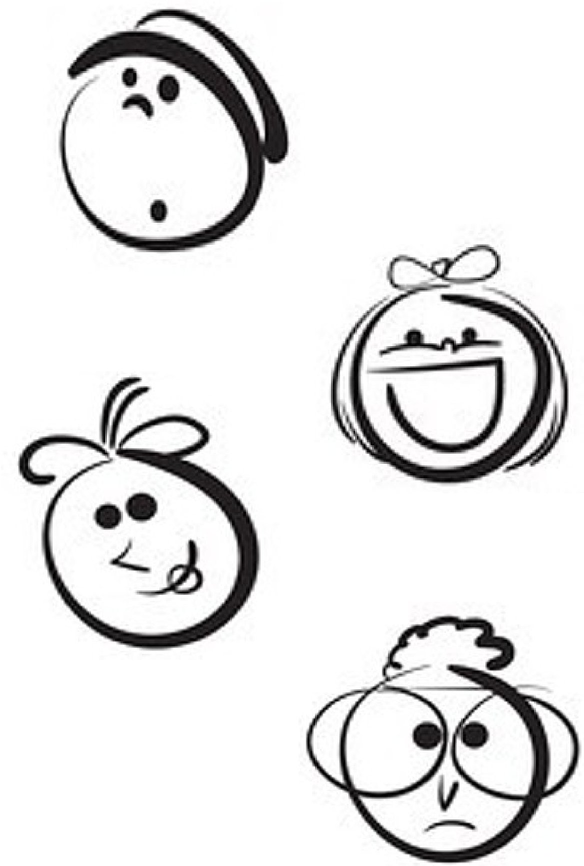
- Motivation
- Required notions
- Maxsum Allocations
  - Two Agents
  - Multiple Agents
- Conclusion

Notions

Cake
[0,1]

Notions

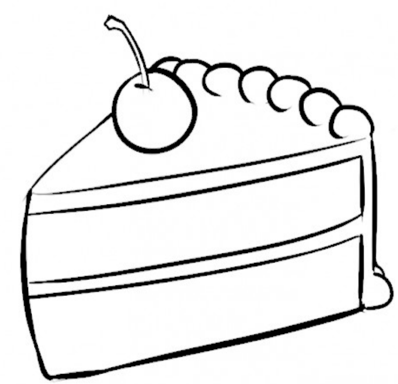
Cake
[0,1]



Agents
$N = \{1,\ldots,n\}$

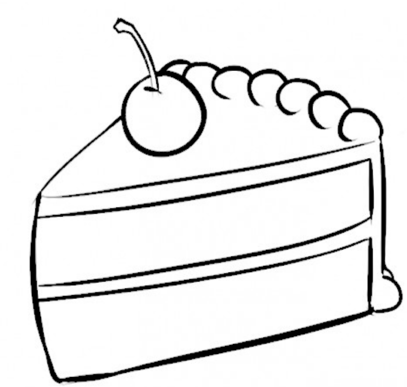
Notions

- Valuation Function: $V_i(X)$



Notions

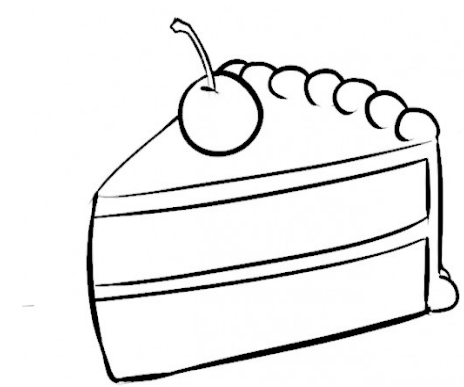
- Valuation Function: $V_i(X) = \sum_{I \in X} \int_I v_i(x)\,dx$



Notions

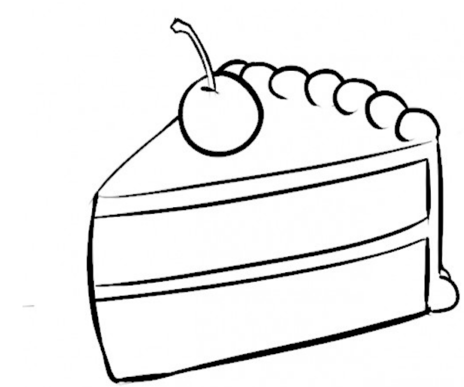
- Valuation Function: $V_i(X) = \sum_{I \in X} \int_I v_i(x)\, dx$

- Normalized: $V_i([0,1]) = 1 \; \forall i \in N$

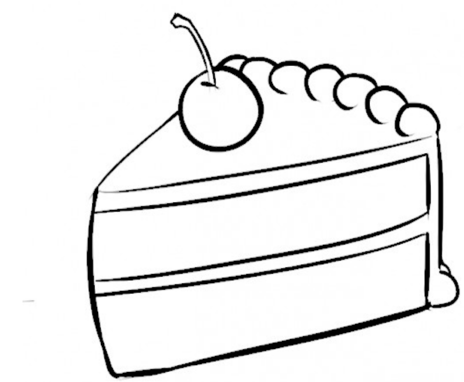


Notions

- Valuation Function: $V_i(X) = \sum_{I \in X} \int_I v_i(x)\,dx$
- Normalized: $V_i([0,1]) = 1 \; \forall i \in N$
- Allocation: $A = (A_1, \ldots, A_n)$

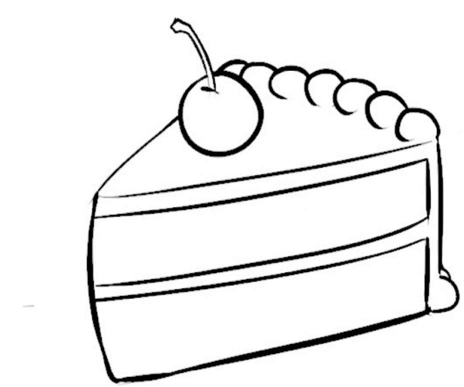


Allocation Fairness



Allocation Fairness

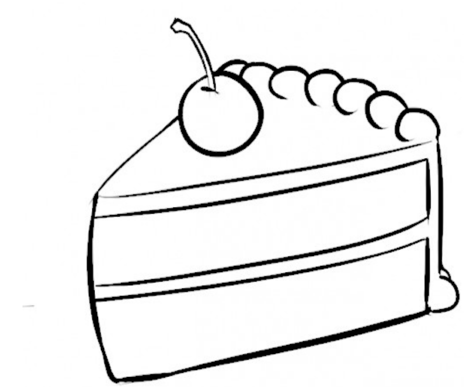
- Envy-Free (EF) : $V_i(A_i) \geq V_i(A_j) \forall\, i, j \in N$



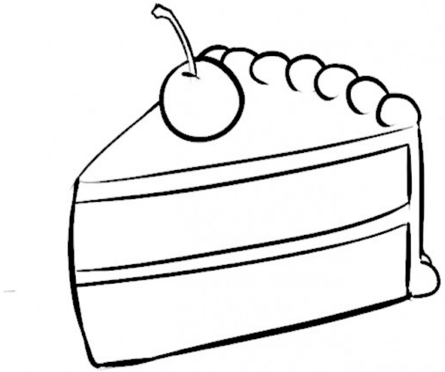
Allocation Fairness

- Envy-Free (EF) : $V_i(A_i) \geq V_i(A_j) \forall i, j \in N$

- Equitable (EQ): $V_i(A_i) = V_j(A_j) \forall i, j \in N$

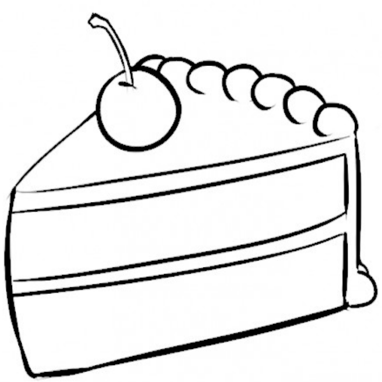


Pareto Optimality (PO)



Pareto Optimality (PO)

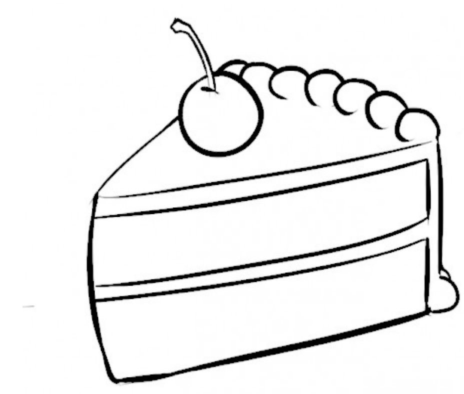
$A = (A_1, \ldots, A_n)$ *Pareto dominated by* $A' = (A'_1, \ldots, A'_n)$



Pareto Optimality (PO)

$$A = (A_1, \ldots, A_n) \; Pareto \; dominated \; by \; A' = (A'_1, \ldots, A'_n)$$

$$V_i(A'_i) \geq V_i(A_i) \; \forall \, i \in N$$

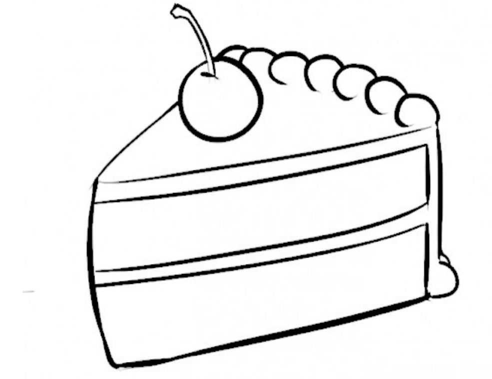


Pareto Optimality (PO)

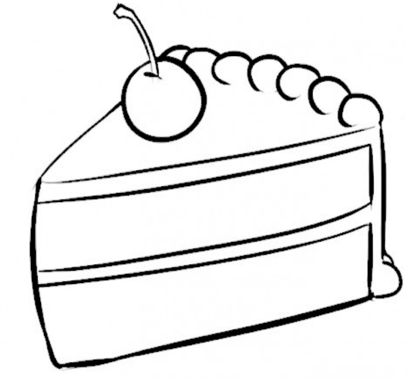
$$A = (A_1, \dots, A_n) \text{ Pareto dominated by } A' = (A'_1, \dots, A'_n)$$

$$V_i(A'_i) \geq V_i(A_i) \; \forall i \in N$$

$$\exists i \in N : V_i(A'_i) > V_i(A'_i)$$

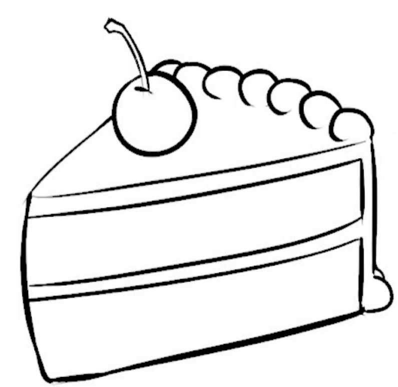


Maxsum Allocations



Maxsum Allocations

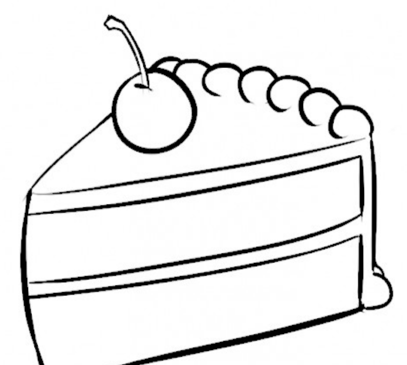
- Social welfare: $sw(A) = \sum_{i=1}^{N} V_i(A_i)$



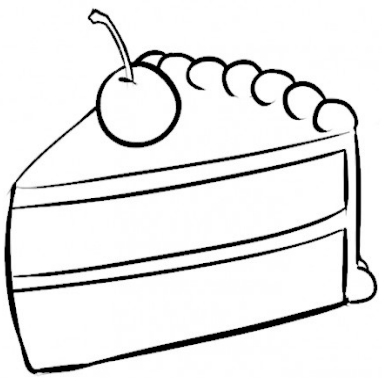
Maxsum Allocations

- Social welfare: $sw(A) = \sum_{i=1}^{N} V_i(A_i)$

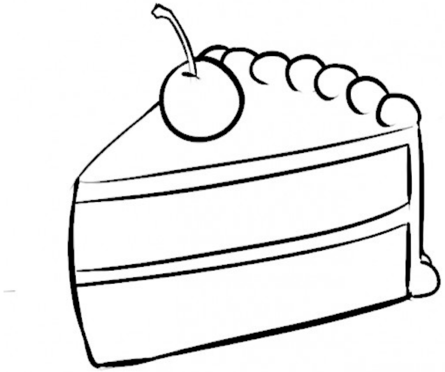
- Allocation *A* maxsum, if social welfare is maximized



Pareto Optimality
Of Maxsum Allocations
(Two Agents)

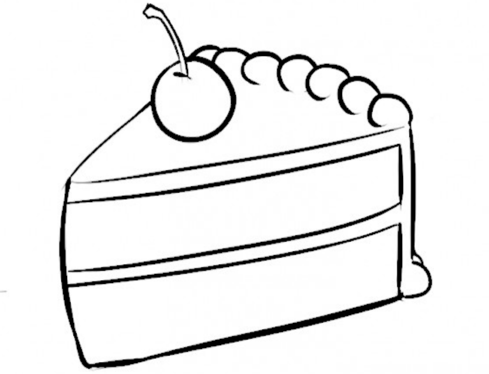


Ratio-based Allocations



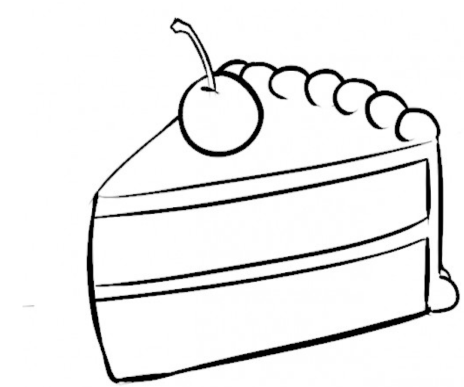
Ratio-based Allocations

- Receiving Agent



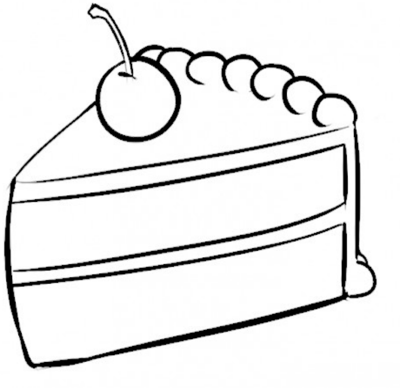
Ratio-based Allocations

- Receiving Agent
- Critical Ratio



PO of ratio-based allocations

Let $A=(A_1, A_2)$ be a ratio-based allocation with agent 1 as the receiving agent s.t. $v=V_1(A_1) \geq V_1(Y_{1\geq 2})$. It holds that



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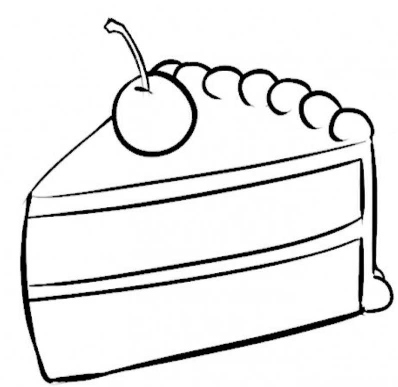
$$V_1(A'_1)=v \Rightarrow sw(A)\geq sw(A')$$

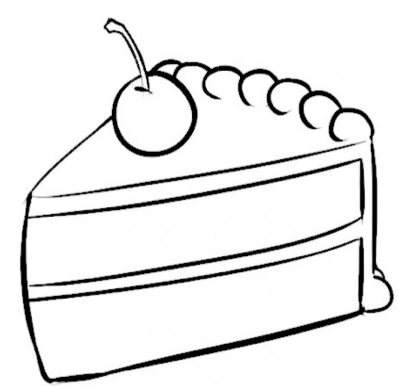


PO of ratio-based allocations

Let $A = (A_{1,} A_2)$ be a ratio-based allocation with agent 1 as the receiving agent s.t. $v = V_1(A_1) \geq V_1(Y_{1\geq 2})$. It holds that

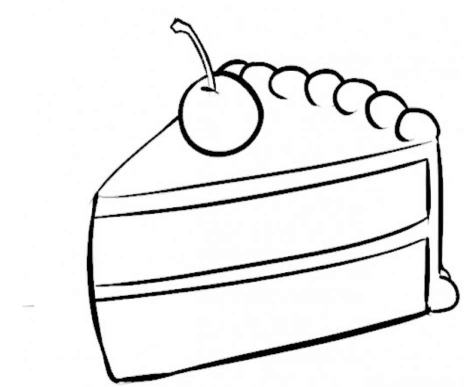
$$V_1(A'_1) = v \Rightarrow sw(A) \geq sw(A')$$

$$V_1(A'_1) > v \Rightarrow sw(A) > sw(A')$$

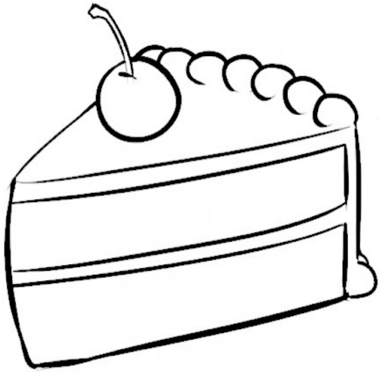


Lemma

Every maxsum EF allocation allocates all intervals, that are desired by some agent.



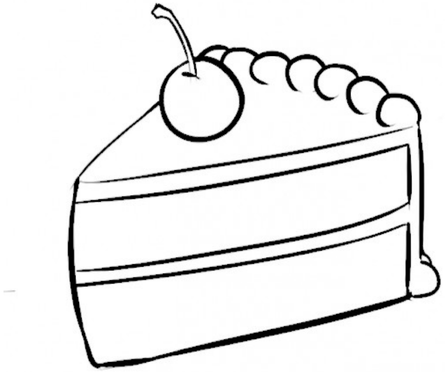
Maxsum EF is PO



Maxsum EF is PO

EF Allocation A, maxsum among all Allocations

There is **no** EF Allocation A, which is maxsum among all Allocations



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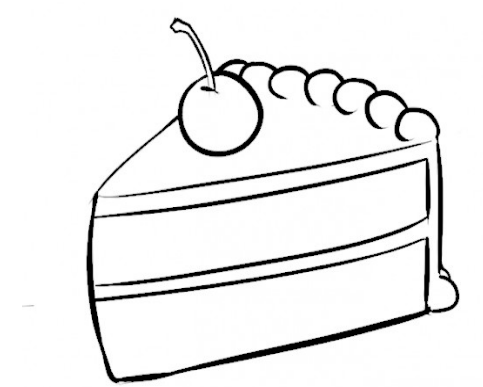

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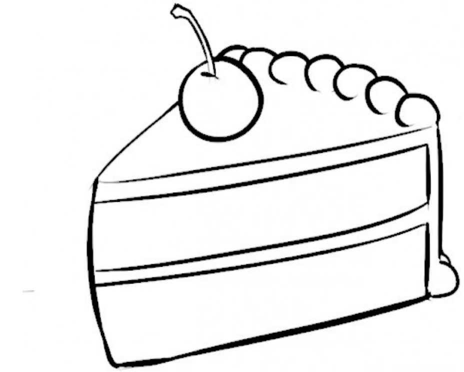
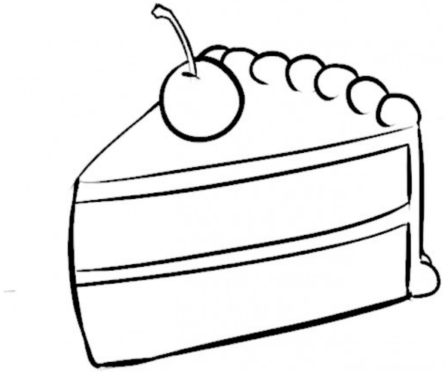


Maxsum EF is PO

There is **no** EF Allocation A, which is maxsum among all Allocations

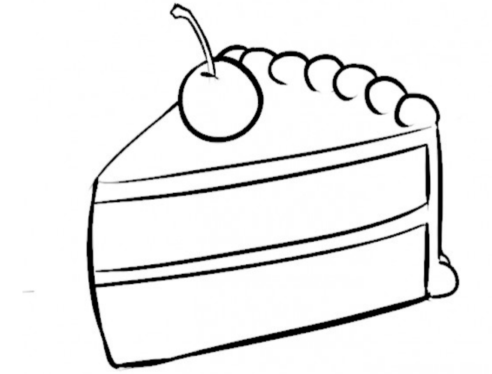
- Assume $V_1\left(Y_{1\geq 2}\right) < 1/2$



Maxsum EF is PO

There is **no** EF Allocation A, which is maxsum among all Allocations

- Assume $V_1\left(Y_{1\geq 2}\right) < 1/2$
- Build ratio-based *A* with $V_1\left(A_1\right) = 1/2$ that is maxsum EF



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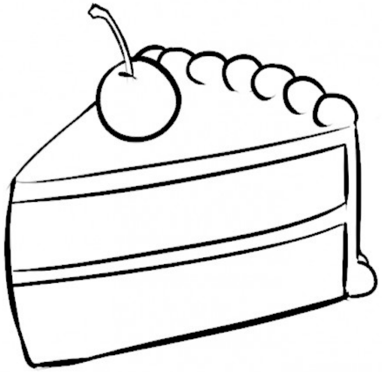
$\Rightarrow$ *A* is PO



Maxsum EF is PO

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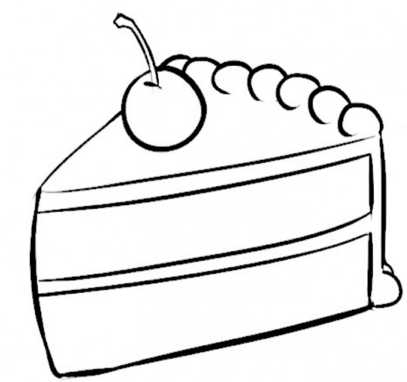
- Assume $V_1\left(Y_{1\geq 2}\right) < 1/2$
- Build ratio-based *A* with $V_1\left(A_1\right) = 1/2$ that is maxsum EF

  $\Rightarrow$ *A* is PO

- Let *A'* be another maxsum EF Allocation



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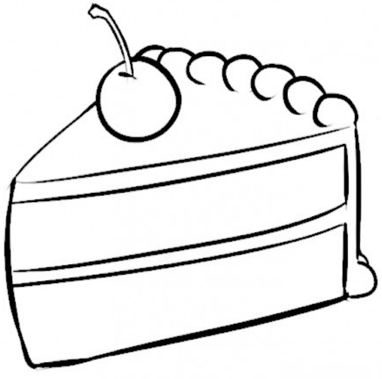
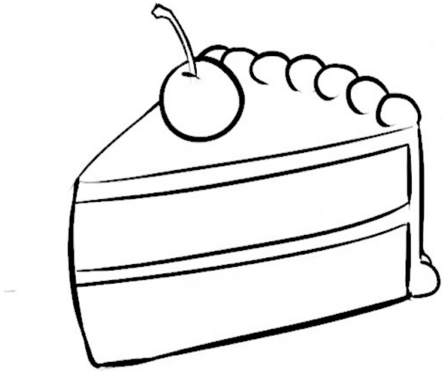


Maxsum EF is PO

There is **no** EF Allocation A, which is maxsum among all Allocations

$$V_1(A'_1) = 1/2$$



Maxsum EF is PO

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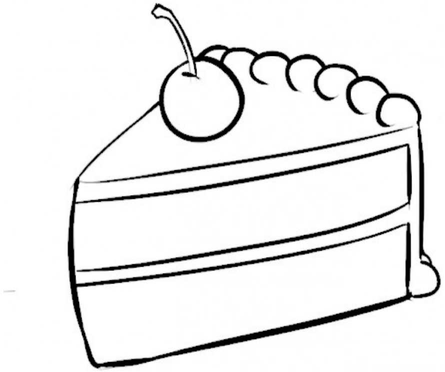
$V_1(A'_1) = 1/2$

$V_1(A'_1) < 1/2$



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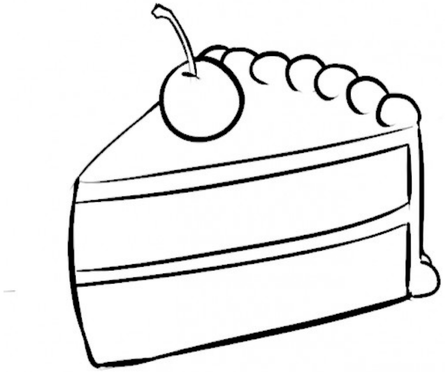
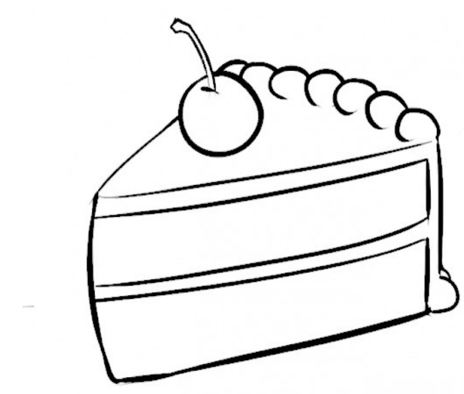


Maxsum EQ is PO

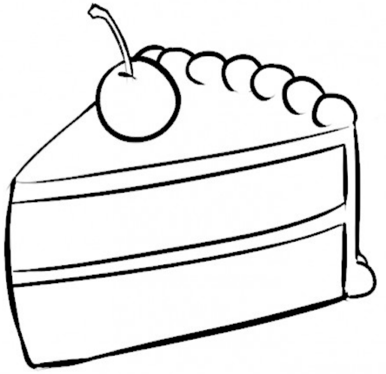


Maxsum EQ is PO

$V_1(Y_{1\geq 2}) \geq V_2(Y_{2>1})$

and vice versa

$V_1(Y_{1\geq 2}) < V_2(Y_{2>1})$



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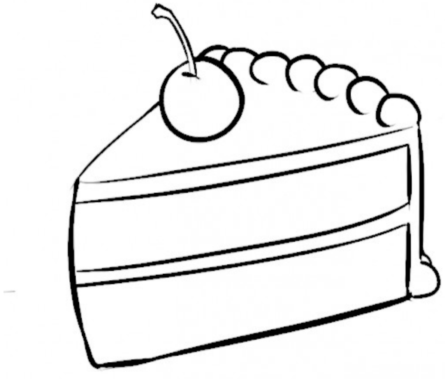
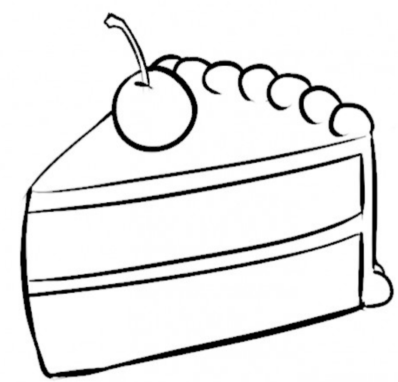


Maxsum EQ is PO

$$V_1\left(Y_{1\geq 2}\right) \geq V_2\left(Y_{2>1}\right)$$

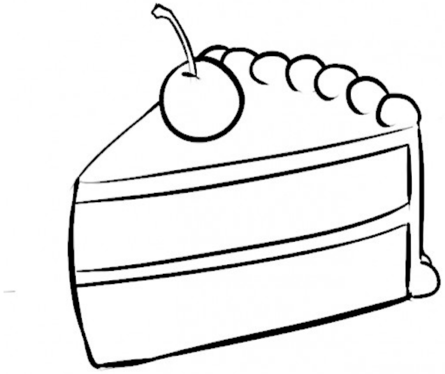
and vice versa

- Allocate $Y_{1>2}$ to agent 1 and $Y_{1>2}$ to agent 2
- Split $Y_{1=2}$



Maxsum EQ is PO

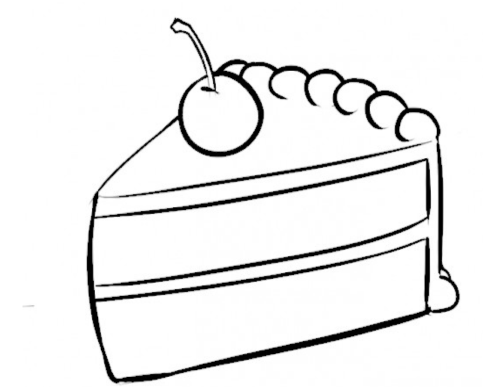
$$V_1\left(Y_{1\geq 2}\right) < V_2\left(Y_{2>1}\right)$$



Maxsum EQ is PO

$$V_1(Y_{1\geq 2}) < V_2(Y_{2>1})$$

Ratio: 1 → 0

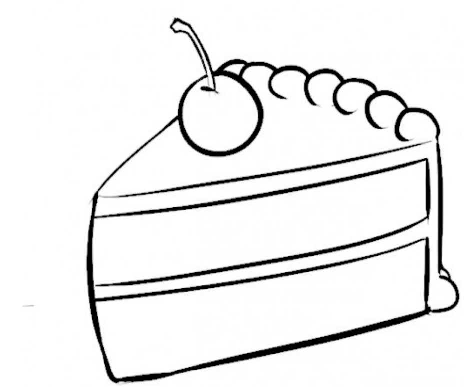


Maxsum EQ is PO

$V_1(Y_{1\geq 2}) < V_2(Y_{2>1})$

Ratio: 1 → 0

$V_1(Y_{1\geq 2})$ → 1



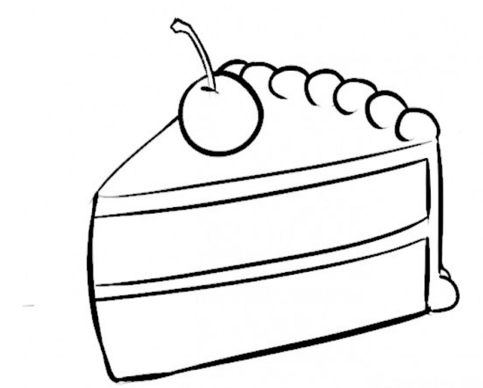
Maxsum EQ is PO

$$V_1\left(Y_{1\geq 2}\right) < V_2\left(Y_{2>1}\right)$$

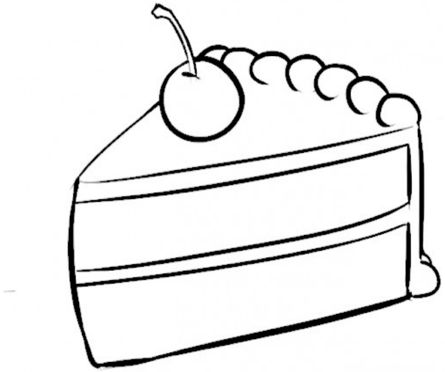
Ratio: 1 → 0

$V_1\left(Y_{1\geq 2}\right)$ → 1

$V_2\left(Y_{2>1}\right)$ → 0



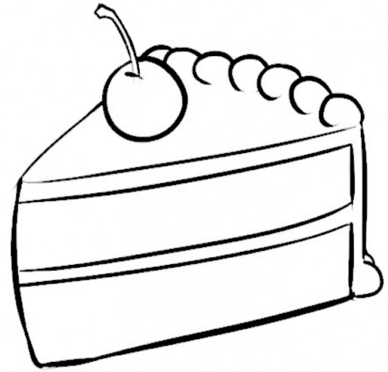
Maxsum EF+EQ is PO



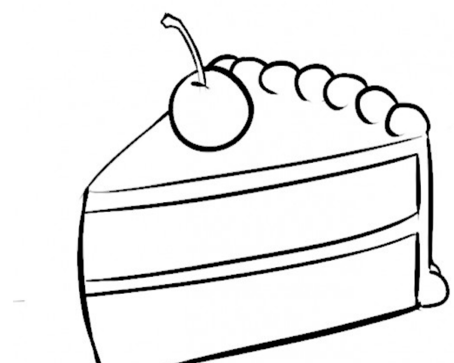
Maxsum EF+EQ is PO

- In maxsum EQ both receive at least value ½

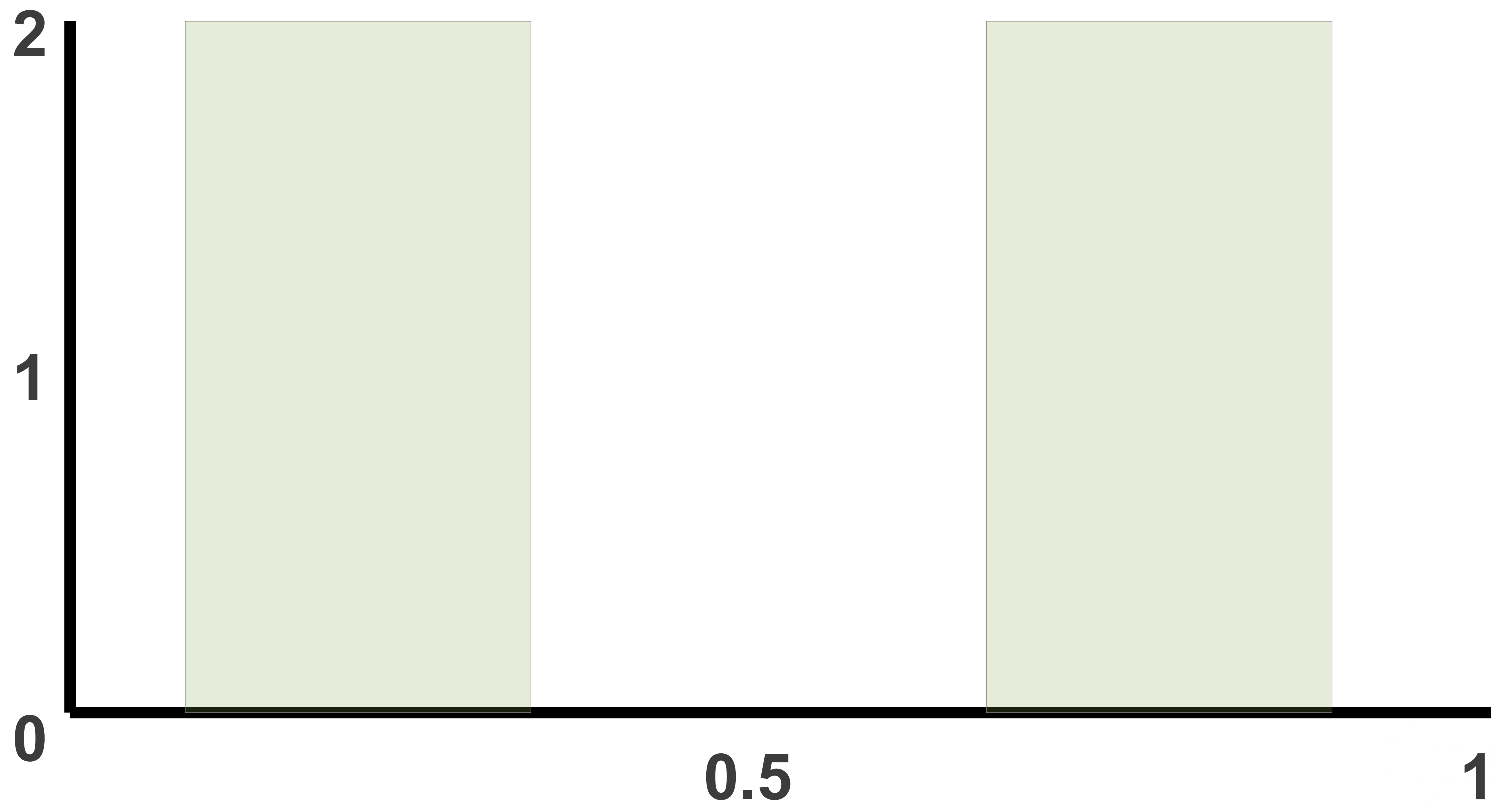
  ⇒ Also EF

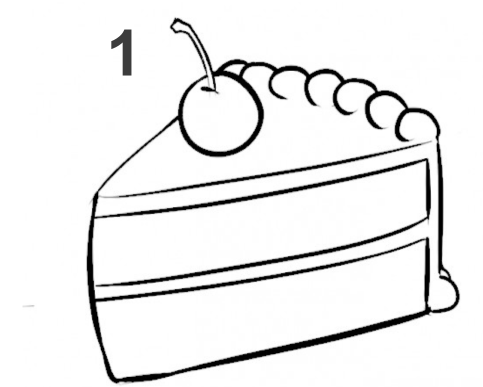


Pareto Optimality
Of Maxsum Allocations
(Multiple Agents)

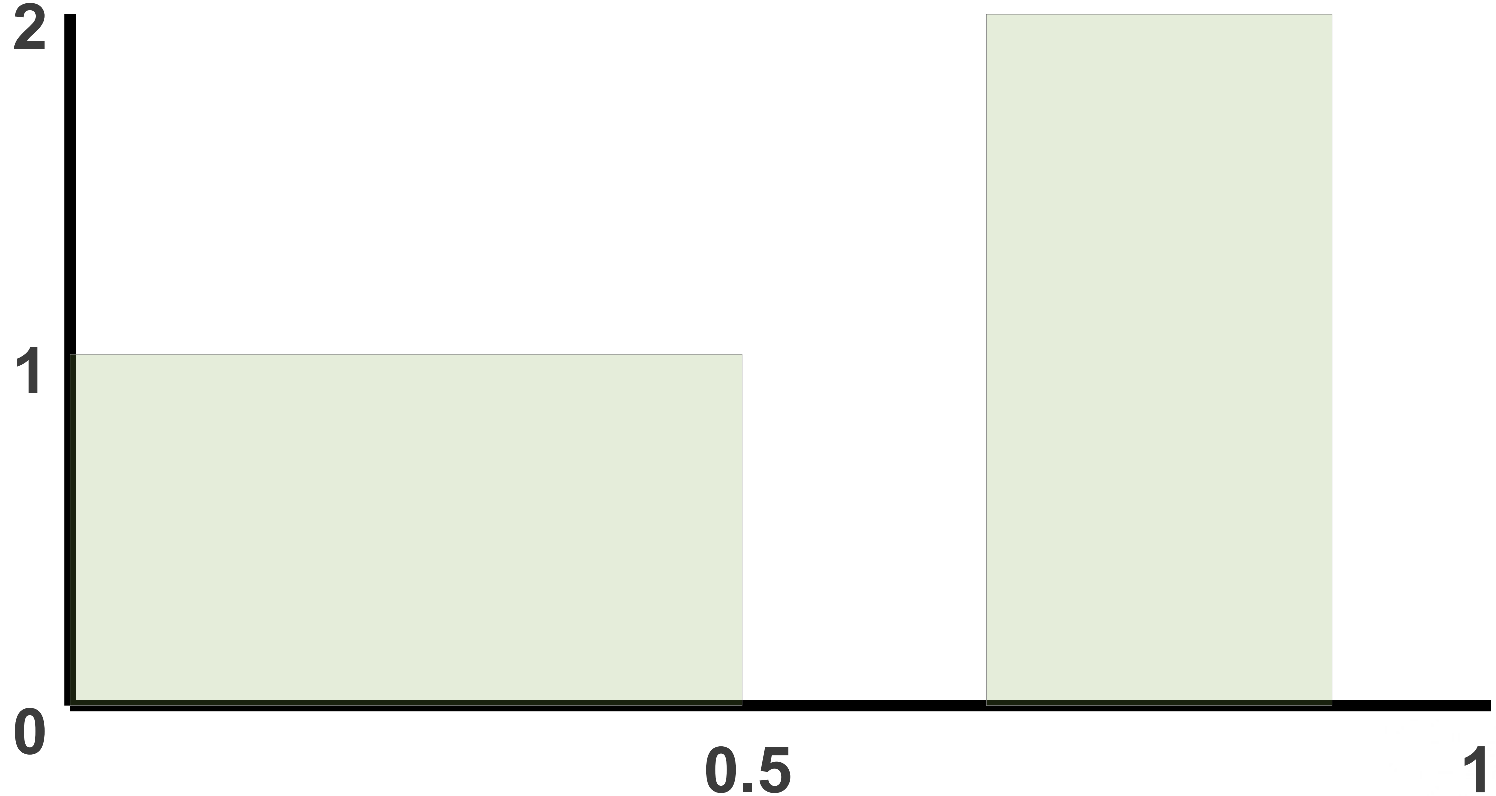


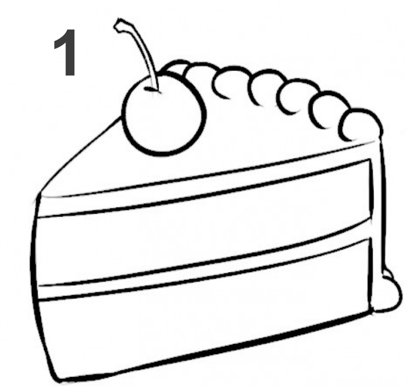
Piecewise uniform valuations (PUV)



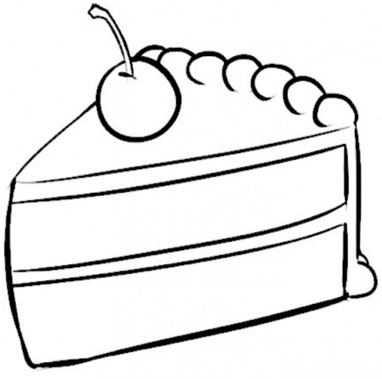


Piecewise constant valuations (PCV)



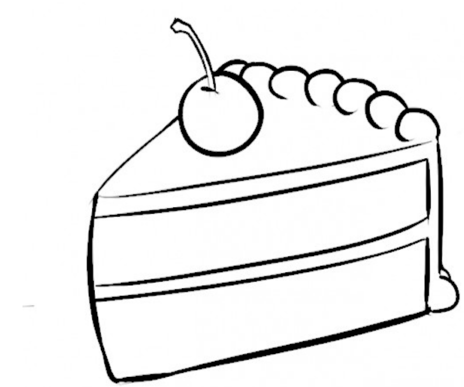


Maxsum EF is PO for PUV

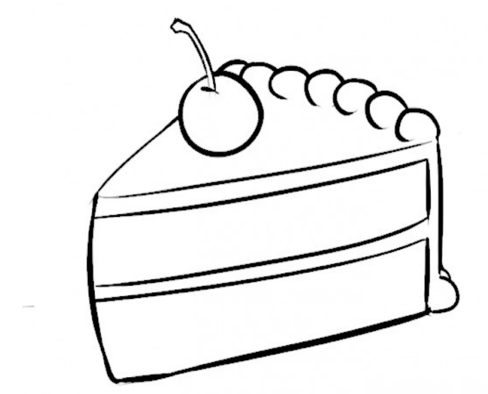


Maxsum EF is PO for PUV

- Each desired interval allocated to some agent sufficient for PO

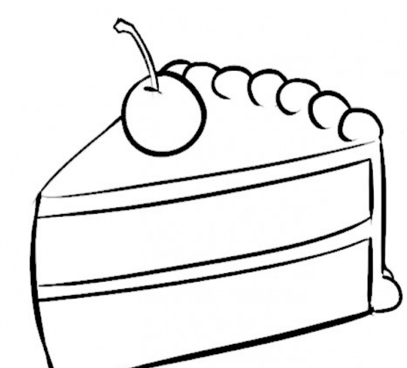


Maxsum EF is PO for PUV



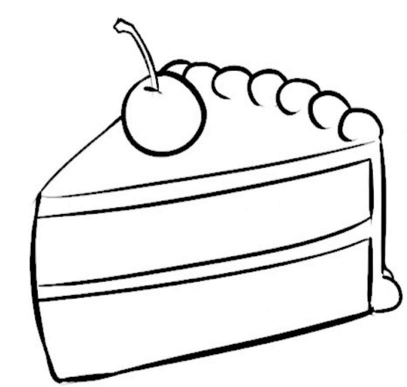
Maxsum EF is PO for PUV

- Assume maxsum EF allocation $A = (A_1, \ldots, A_n)$



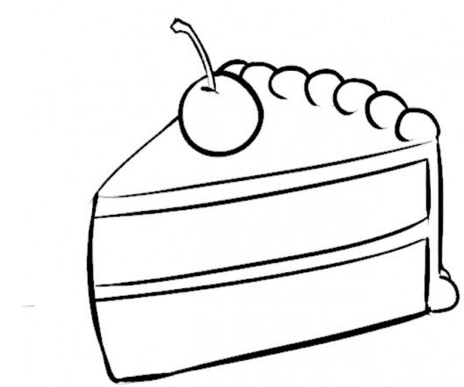
Maxsum EF is PO for PUV

- Assume maxsum EF allocation $A = (A_1, \ldots, A_n)$
- X' not allocated

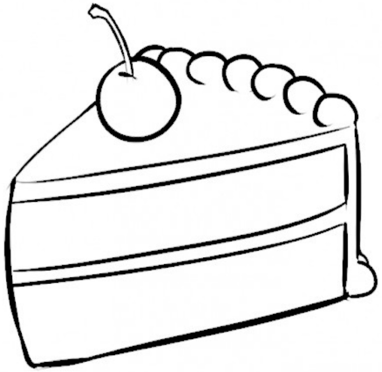


Maxsum EF is PO for PUV

- Assume maxsum EF allocation $A = (A_1, \ldots, A_n)$
- X' not allocated
  - ➔ Split
  - ➔ Give equal share to each agent



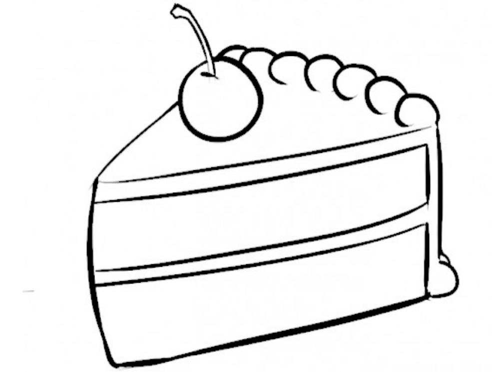
Maxsum EQ not PO for PUV



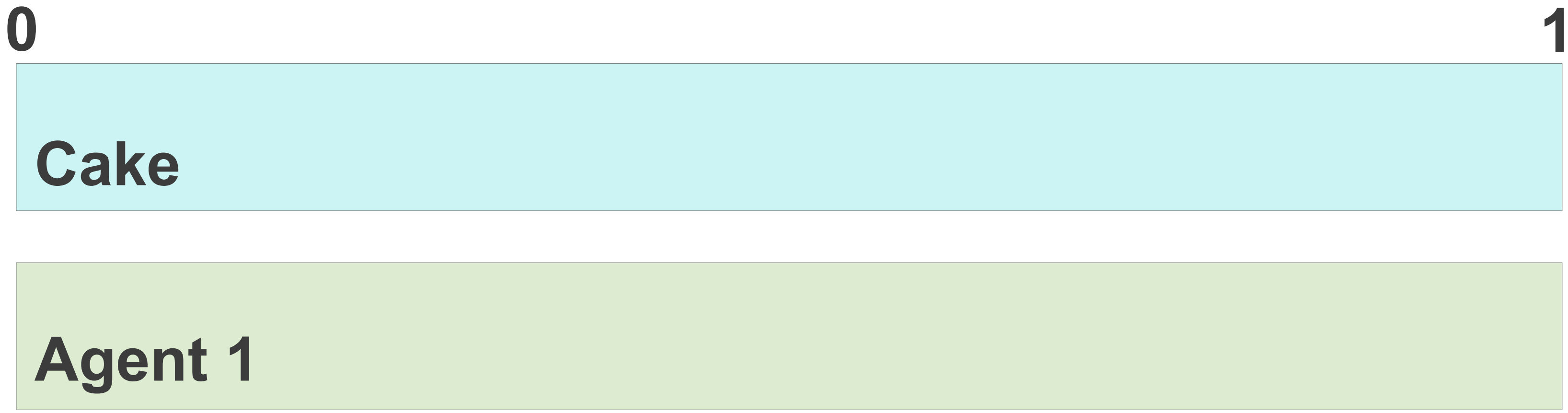
Maxsum EQ not PO for PUV

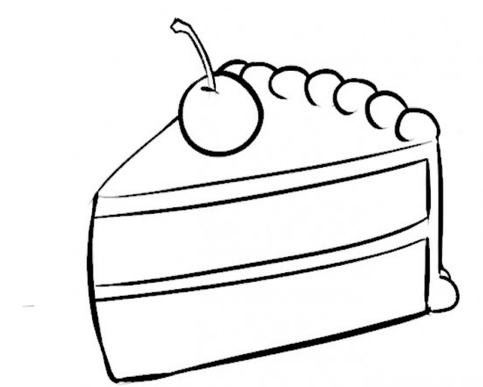
0 1

Cake

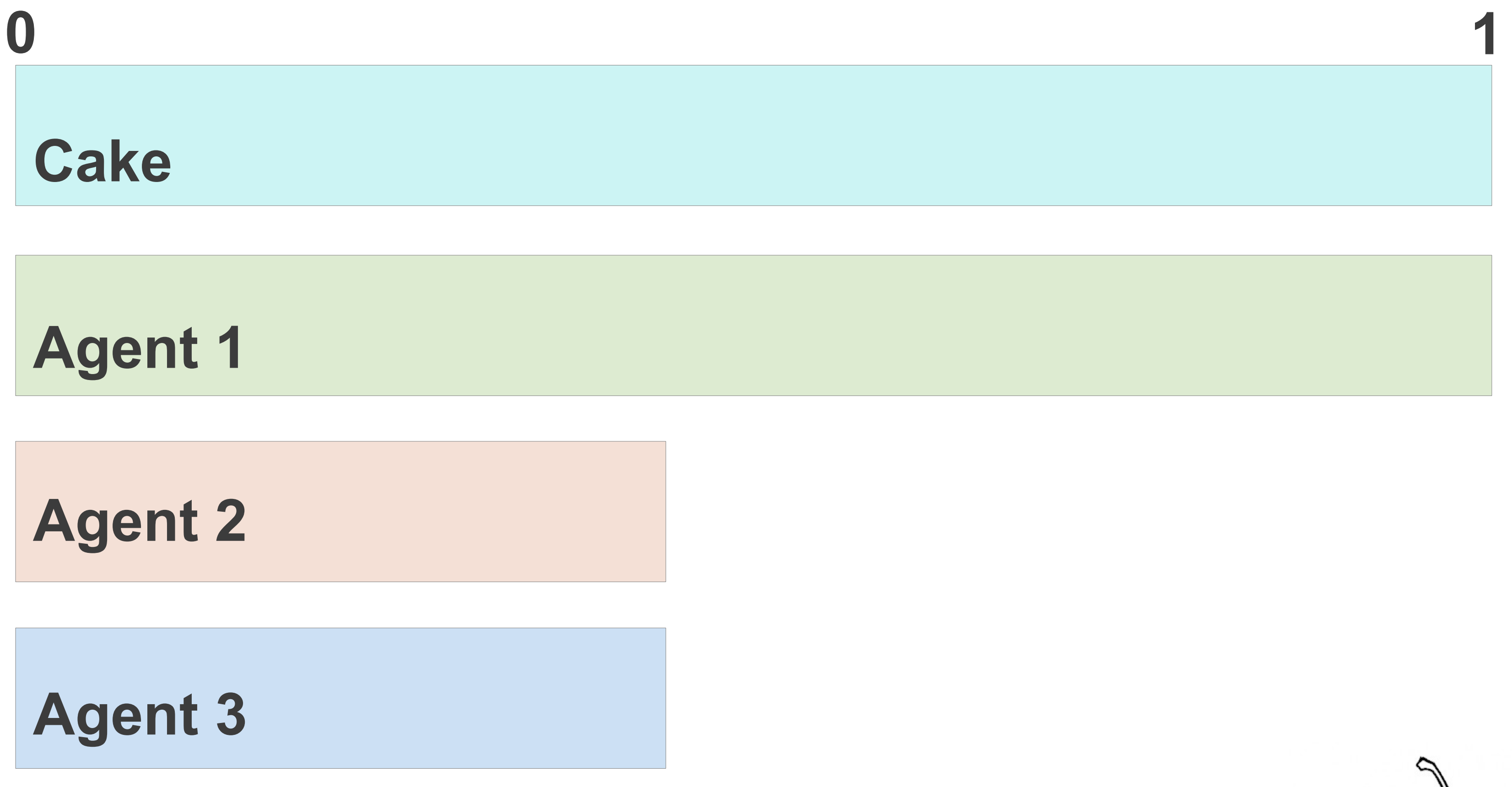


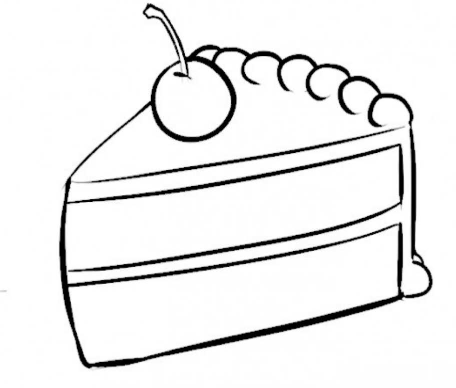
Maxsum EQ not PO for PUV



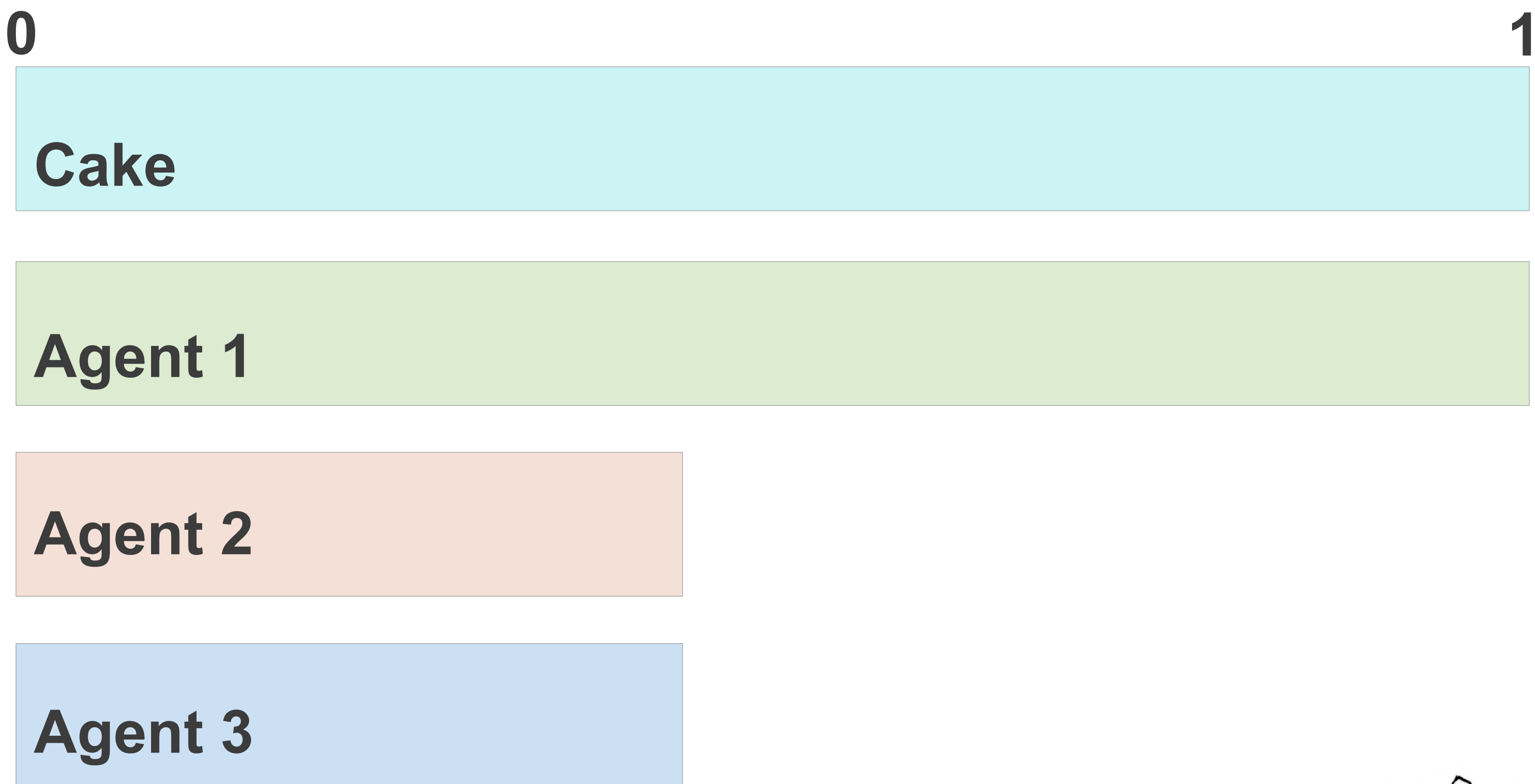


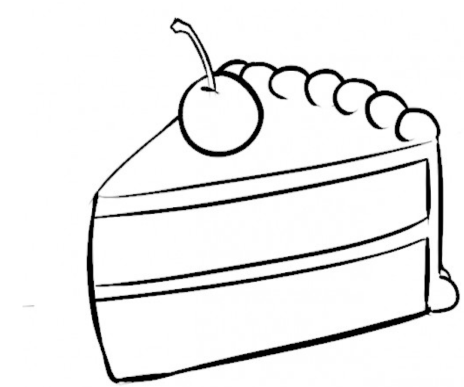
Maxsum EQ not PO for PUV



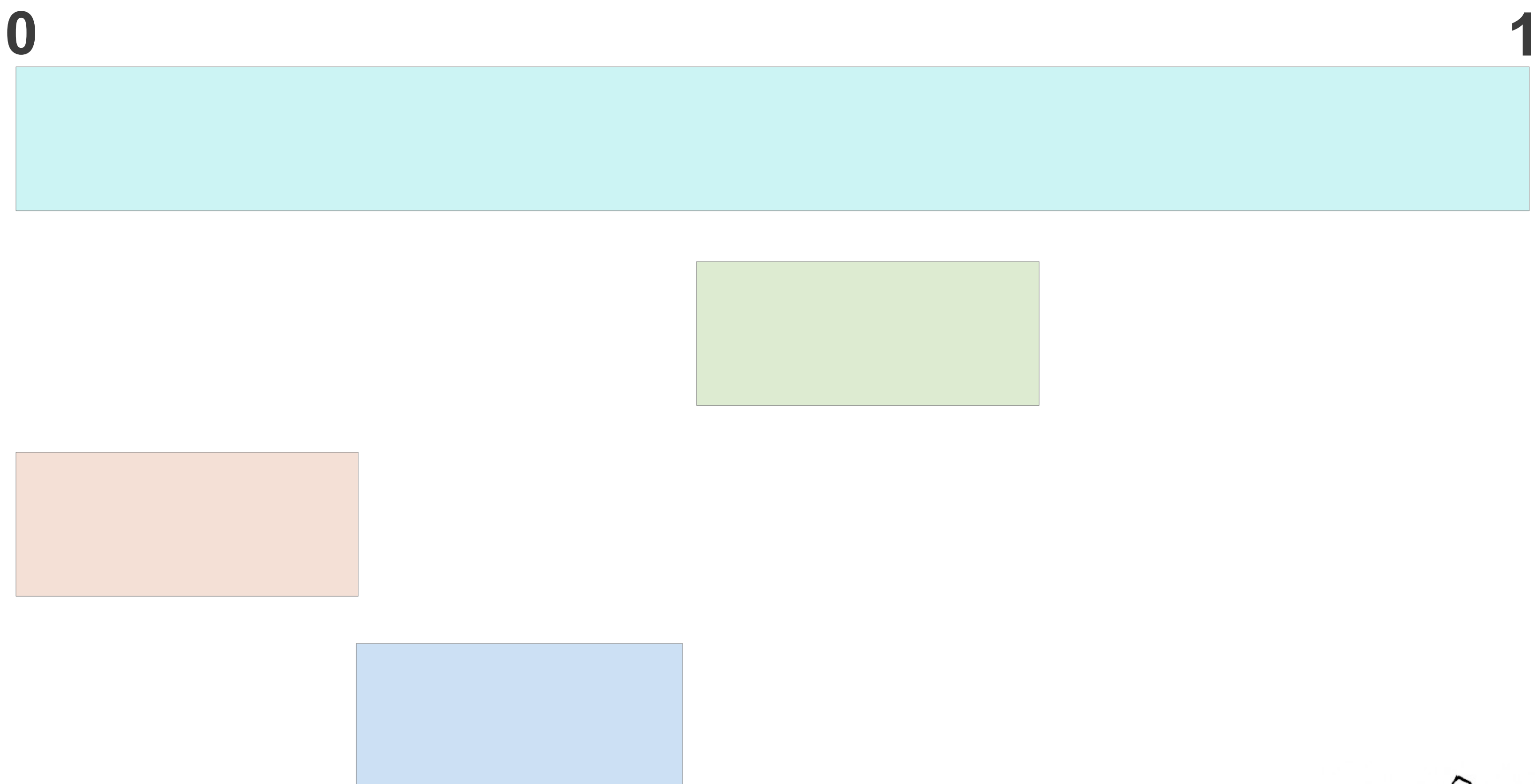


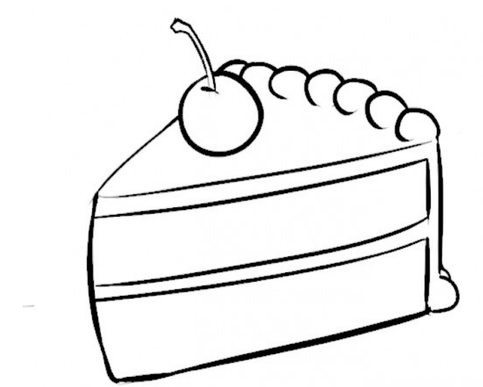
Maxsum EQ not PO for PUV



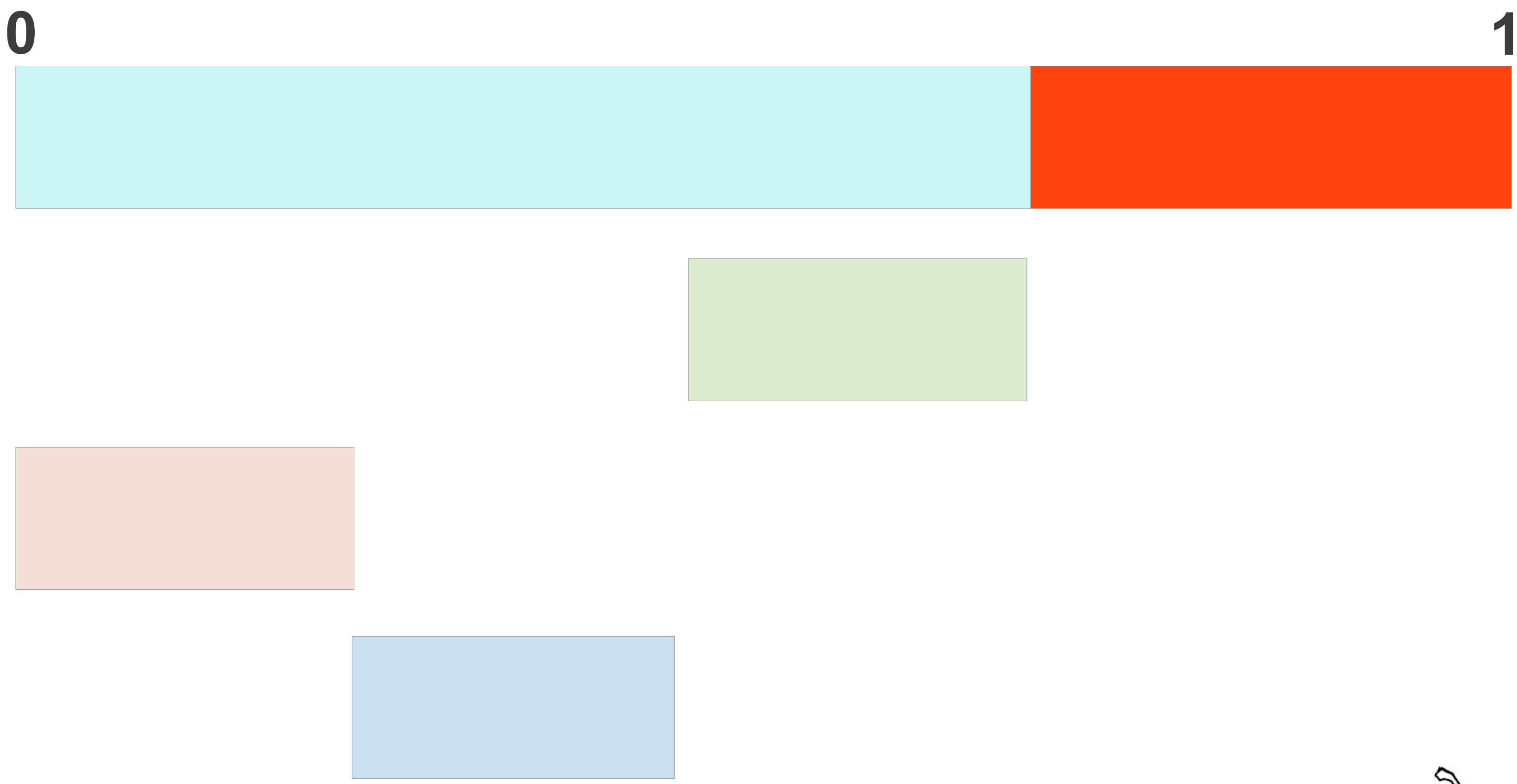


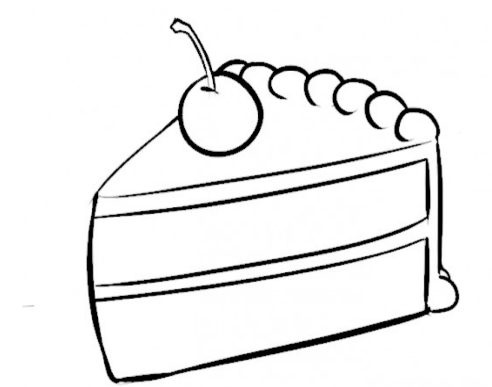
Maxsum EQ not PO for PUV



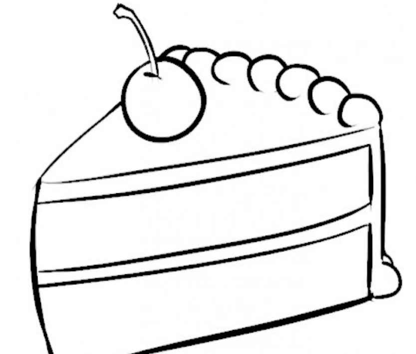


Maxsum EQ not PO for PUV

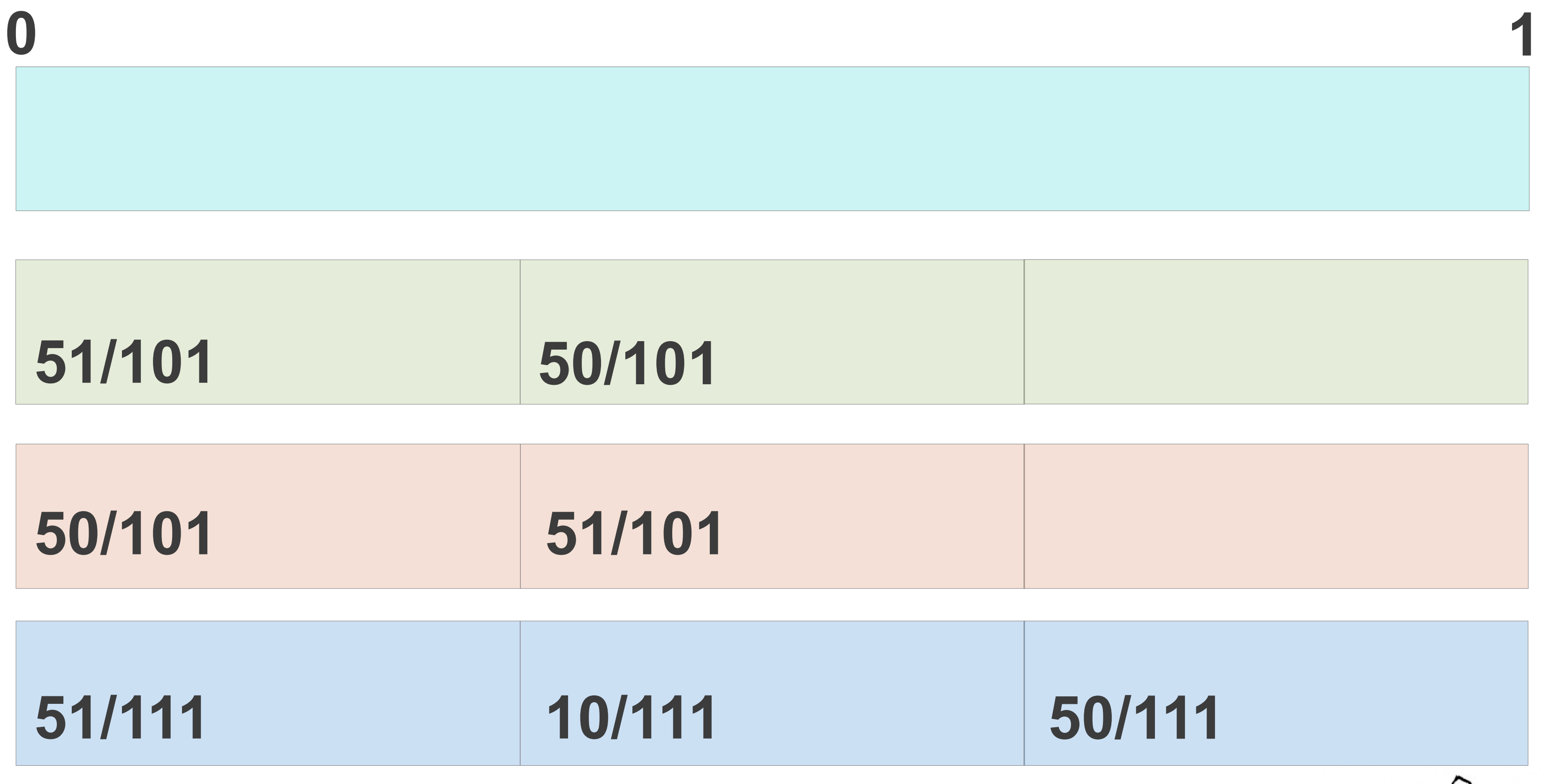


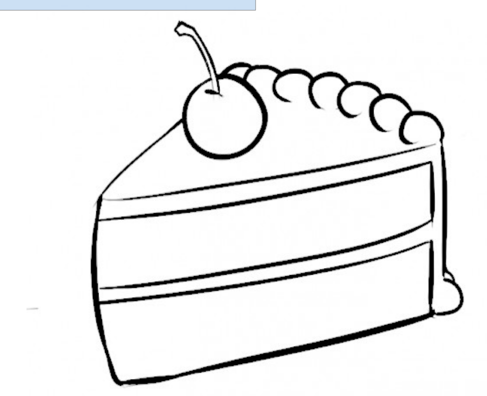


Maxsum EF not PO for PCV

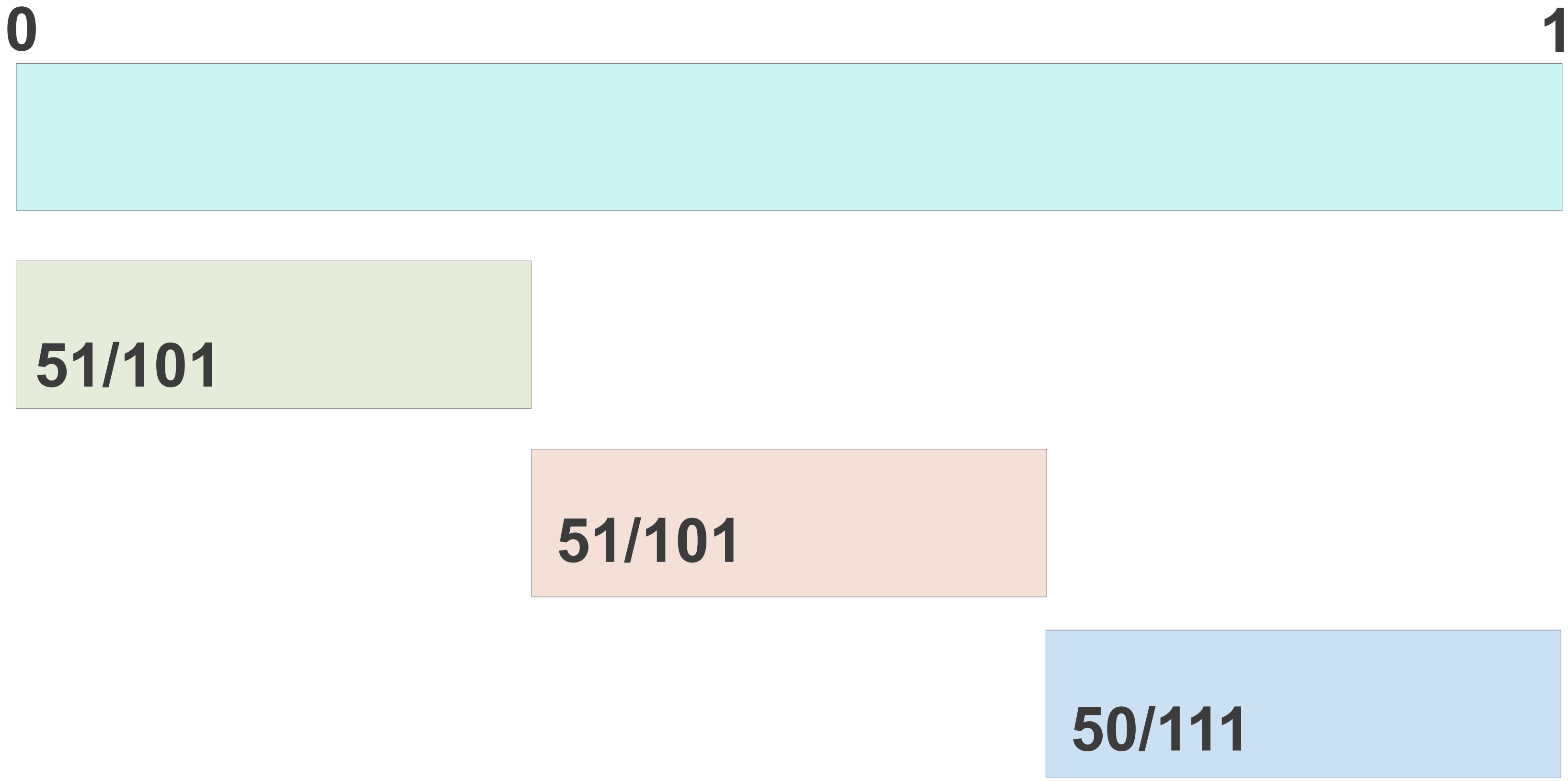


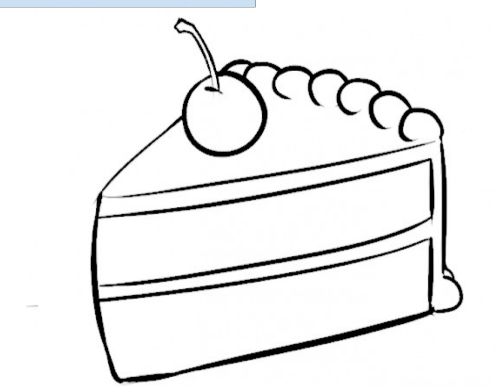
Maxsum EF not PO for PCV



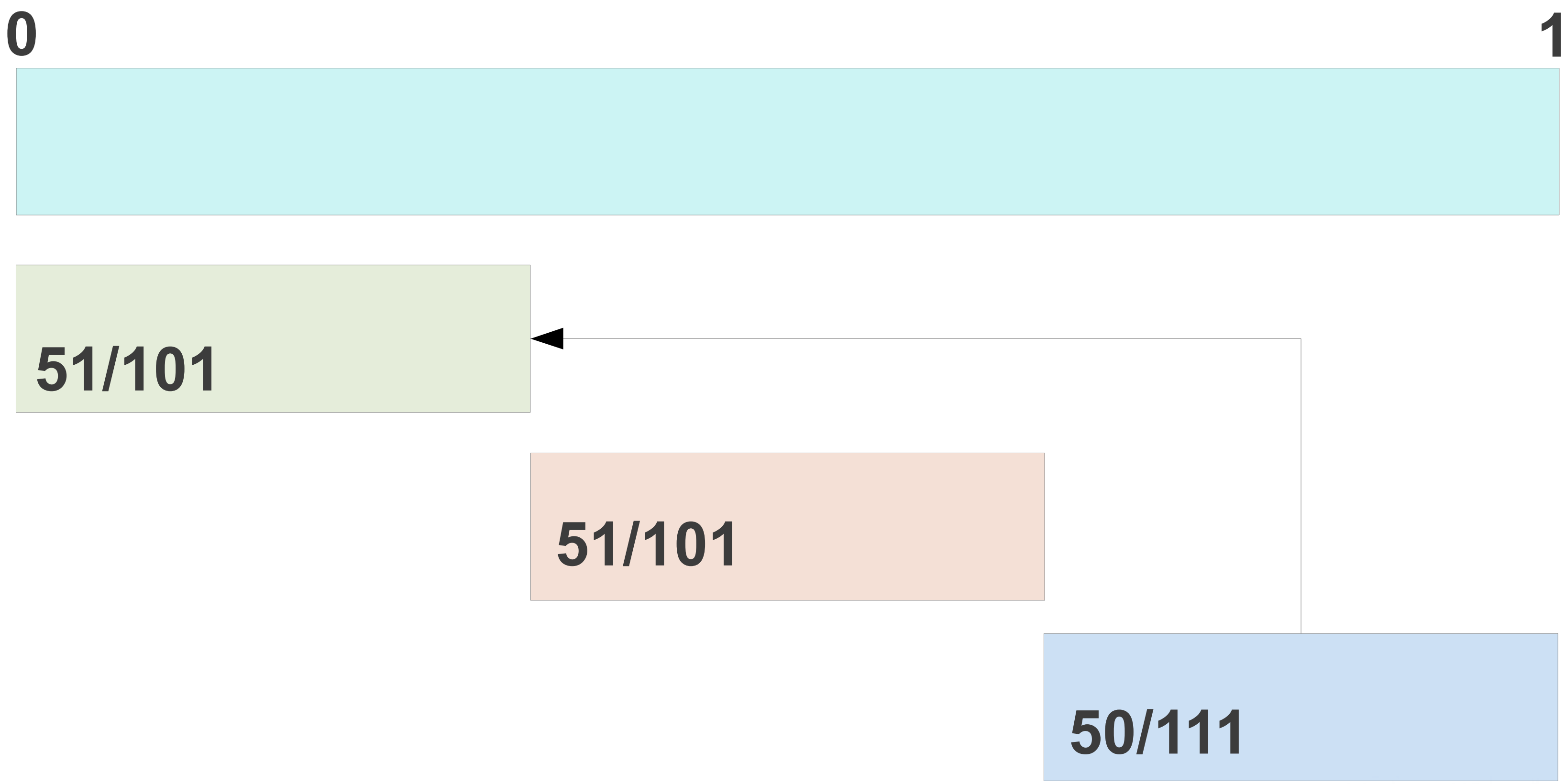


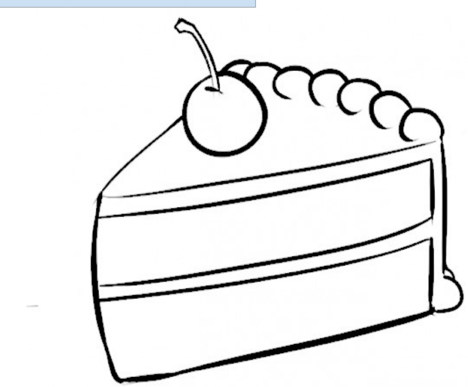
Maxsum EF not PO for PCV



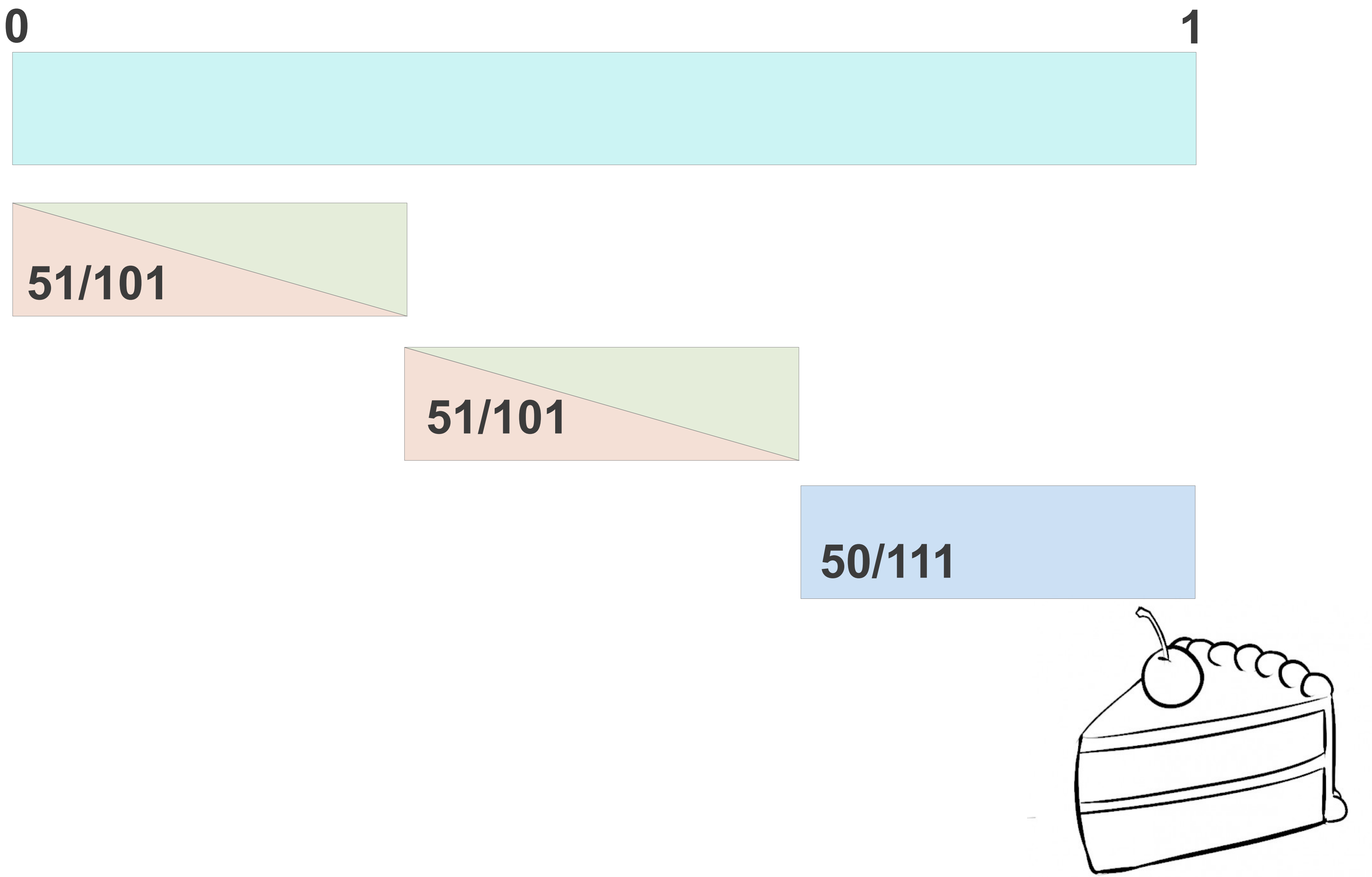


Maxsum EF not PO for PCV





Maxsum EF not PO for PCV



Conclusion

Conclusion

- Two Agents
  - ➔ EF, EQ, EF+EQ are PO

Conclusion

- Two Agents
  - ➔ EF, EQ, EF+EQ are PO
- Multiple Agents
  - ➔ EF is PO under PUV
  - ➔ EQ, EF+EQ not PO under PUV
  - ➔ EF not PO under PCV

Questions?