Basic Mathematical Techniques for Computer Scientists

Propositional Logic

October 22, 2012

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- Mathematical proof
  - Propositions
    - Statements which are either true, or false
  - Axioms
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  - Logical deductions
    - Ways to combine true propositions to get more such

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    - Loosely put: “self-evident” truths
  - Logical deductions
    - Ways to combine true propositions to get more such
- Definition: A mathematical proof of a proposition is
  - A sequence of *logical deductions*
  - Starting from a set of *axioms*
  - And leading to the proposition

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- Some more examples:
  - Every tree has a leaf.
  - Any pair of lines in the plane intersect.
  - $n^2 + n + 41$ is a prime number.
- Propositions can be combined to form other propositions.

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- "AND" : *some proposition* AND *some proposition*
- Written $\wedge$ for short
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- Note:
  - The word "and" has many meanings in English . . .
  - (Wiktionary lists around fifteen.)
  - . . . of which exactly one is valid in propositional logic.

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Table: The truth table for "AND".

P	Q	$P \wedge Q$
False	False	False
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- Note:
  - The word "or" has many meanings in English ...
  - ... of which exactly one is valid in propositional logic.
- Beware, in particular, about the following uses:
  - The *exclusive or*: "Coffee or tea?"
  - *Otherwise*: "Hurry or you will miss the bus!"
- The logical "or" is neither of these.

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Negation

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- Written $\neg$ for short
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  - $n^2 + n + 41$ is a prime number.
  - L_1, L_2 intersect in the plane.
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- A *predicate* is a proposition whose truth (or falsity) *depends* on the value of one or more variables.
  - A predicate can be thought of as a function ...

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- $\forall n \in \mathbb{R}\ \ n \geq 2 \implies n^2 \geq 4$
  - Read as: "... *if* $n \geq 2$ is true, *then* $n^2 \geq 4$ is true."
  - Is the above *implication* true?

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  - $\mathrm{A} \implies \mathrm{B}$ is *true* in **all** other cases.
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    - This is counter-intuitive.

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  - $A \implies B$ is *true* in **all** other cases.
    - Including when A is *false* and B is *true*.
    - This is counter-intuitive.
  - For instance, the following is *true*:
    - If $2 = 3$, then I am the King of France.

More examples

- $\forall x, y \in \mathbb{R} \; x > y \implies x^2 > y^2$

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- $\forall x, y, z \in \mathbb{N}\ (x < y) \wedge (y < z) \implies (x < z)$
- Exercise: Draw the truth table for A $\implies$ B.

Equivalence

- A $\iff$ B
- Read as "A if and only if B".
- Sometimes written as "A iff B".
  - Note the non-standard word.
  - Introduced by Halmos.
- A $\iff$ B is true when
  - A $\implies$ B is true, AND
  - B $\implies$ A is true.
- It is *false* in *all* other cases.

Thank You!