Basic Mathematical Techniques for Computer Scientists

Introduction to Discrete Probability, Part II

January 28, 2013

Recap

- Experiment, outcome, event
- Many examples of computing probabilities
  - When all outcomes are equally likely
  - Involves counting subsets
  - Very useful: Ways of counting things
- When all outcomes are equally likely
  - The probability of an event:

$$\frac{\text{Size of the event}}{\text{Total number of possible outcomes}}$$

- Random choices, populations, sampling with/without replacement

Recap

- Formal framework:
  - Sample space, events, and probability functions
  - Axioms of discrete probability:
    - Laws which a probability function must obey
  - Probability spaces
  - Impossible and certain events

Questions?

Tossing a coin till it turns up Heads

Warm-up

- Experiment:
  - Toss a fair coin repeatedly, ...
  - ...till the *first* time it turns up Heads
  - Outcome: the *number* of tosses it takes each time
- What is the sample space Ω?
- What is the probability that the outcome is i?
  - For a fixed $i \in \mathbb{N}$?
- If $S \subseteq \mathbb{N}^+$, then what is $\Pr(S)$?
- If $S = \{2i \mid i \in \mathbb{N}^+\}$, then what is $\Pr(S)$?
- Is this Pr a valid probability function?

Axioms of Discrete Probability

Recap

A function $\Pr : 2^{\Omega} \to \mathbb{R}$ defined on a sample space Ω is said to be a probability function if and only if it satisfies the following axioms:

1. **(Non-negativity)** For each $S \subseteq \Omega$, $\Pr(S) \geq 0$
2. **(Normalization)** $\Pr(\Omega) = 1$
3. **(Additivity)** If $S_1, S_2, \ldots, S_n$ are pairwise disjoint events, then

$$\Pr(\bigcup_{1}^{n} S_i) = \sum_{i=1}^{n} \Pr(S_i)$$

Simple applications of the axioms

- $(\Omega, \Pr)$ is a probability space
- A, B are arbitrary events in the space
- Prove:
  - If $A \cap B = \emptyset$, then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$
  - $\Pr(\overline{A}) = 1 - \Pr(A)$
  - $\Pr(A) \leq 1$
  - $A \subseteq B \implies \Pr(B) \geq \Pr(A)$
  - $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

Bell Pepper

Aka sweet pepper, pepper, capsicum

- (For those who didn't get it last time)

Figure : "Traffic lights" pack of bell peppers.

- Picture credit: Luc Viatour / www.Lucnix.be

Stocking bell peppers

The tale of a shop assistant

- A shop assistant in a grocery starts stocking bell peppers on shelves before the shop opens. When he comes to the last shelf, there are 10 bell peppers remaining in his basket, 4 red and 6 green. Since there is plenty of space on the shelf for these, he arranges these 10 peppers in a single row on this shelf.
- If the assistant puts these 10 bell peppers randomly on the shelf, what is the probability that all peppers of at least one colour appear together?

Bell peppers

Good and bad

- When a bell pepper goes bad, sometimes it turns soft. A soft bell pepper is probably not good to eat. So people don't buy these.
- Here are the contents of our friend's basket when he comes to the last shelf on another day:

	Good	Soft	Total
Red	4	3	7
Green	2	6	8
Total	6	9	15

- He picks a bell pepper at random from the basket
  - What is the probability that it is red?
  - When he feels the pepper (before looking at it), he finds that it is soft. What is the probability now that it is

Bell peppers

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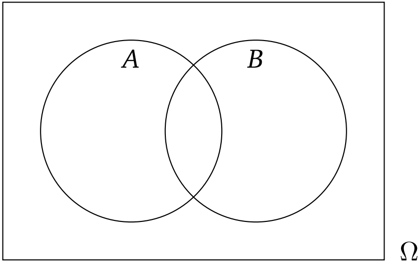
- He picks a bell pepper at random from the basket
  - What is the probability that it is green?
  - When he feels the pepper (before looking at it), he finds that it is soft. What is the probability now that it is green?

More information affects probabilities

- Extra information: the picked pepper is soft
- This changed the probabilities of the pepper being of each colour
- In general: Additional information changes probabilities
- A common example:
  - When two equally strong football teams play each other
    - Each has an even chance of not losing
    - (One could take this as the *definition* of “equally strong”!)
  - If one player of team A is *sent off* with a red card in the first minute
    - Then the probabilities change drastically!

More information affects probabilities

- Consider a sample space with all outcomes equally likely:



- What is $\Pr(A)$?
    - In terms of the cardinalities of the sets?
- What is $\Pr(A)$, *given that* event B has **already** happened?

Conditional probability

The general case

- Let $(\Omega, \Pr)$ be a probability space, and let A and B be two events in this space. If $\Pr(B) > 0$, then the *conditional probability of A given B* is defined to be

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Conditional probability

Blind coin tosses

- You are blindfolded for these experiments
- You toss three fair coins
  - What is the probability that all tosses are Heads?
  - Somebody tells you that at least one coin turned up Heads
    - Now what is the probability that all tosses are heads?
    - Does this change make sense?

Independent events

In real life

- When are two events in real life “independent”?
  - Examples?
  - Non-examples?
- Two events A and B are said to be independent if
  - The occurrence of one does **not** affect the probability of the other
  - Both ways!

Independent events

In probability theory

- Two events A and B are said to be independent if
  - The occurrence of one does **not** affect the probability of the other
  - (We assume some probability space where A and B live)
- Suppose A and B are a pair of independent events
  - (This means they are independent *with respect to each other* ...
  - ... not that they are independent in some wider sense!)
  - What does this mean in terms of the *probabilities* of A and B?
    - Hint: Think conditional probabilities

Independent events

In probability theory

- If A and B are a pair of independent events,

$$\Pr(A \mid B) = \Pr(A)$$

  and

$$\Pr(B \mid A) = \Pr(B)$$

- The probability that A happens is *independent* of whether B happened or not
  - And *vice versa*
- This leads us to the formal definition of independence

Independent events

In probability theory

- Two events A and B are said to be *independent* if (and only if!)

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

- Two events A and B which are **not** independent are said to be *dependent*
  - Meh!
- Suppose A and B are independent
  - Are A and $\overline{B}$ independent, or dependent?

Independence

Examples

- Pick the top card from a shuffled deck
  - A = {The card is a King}
  - B = {The card is a spade}
  - Are A and B independent?
- Pick the first two cards from a shuffled deck
  - A = {The first card is a King}
  - B = {The second card is a spade}
  - Are A and B independent?
- Pick the first two cards from a shuffled deck
  - A = {The first card is a spade}
  - B = {The second card is a spade}
  - Are A and B independent?

Thank You!