Basic Mathematical Techniques for Computer Scientists

Sets and Relations

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Recap

Questions?

- Some proof techniques
- Direct proof
  - Argue directly from axioms and known theorems
- Proof by contrapositive
  - To prove $\mathrm{P} \implies \mathrm{Q}$ ...
  - ... prove $\neg\mathrm{Q} \implies \neg\mathrm{P}$ **instead.**
    - Contrapositive is **not** the same as Con*verse*
- Proof by contra*diction*
  - To prove P ...
  - ... assume $\neg$P and arrive at a falsity.
- Proof by induction
  - To prove $\forall n \in \mathbb{N}\ \mathrm{P}(n)$:
    1. Prove $\mathrm{P}(0)$
    2. Prove $\forall n \in \mathbb{N}\ (\mathrm{P}(n) \implies \mathrm{P}(n+1))$

Sets

- Set: A *well-defined* collection of objects
  - Things in the collection are *elements* of the set
  - $x \in A$: "Element x *is in* set A"
  - Order does *not* matter
  - Elements have *multiplicity* 1
- Different ways of expressing sets?
- When are two sets said to be *equal*?
  - When are two sets *not* equal?
- The *cardinality* of a set A
  - Denoted $|A|$
- The empty set, $\emptyset$

Sets

- Notation for intervals on the real line:
  - *Closed* interval $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
  - *Open* interval $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$
  - *Half open* interval $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
  - *Half open* interval $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}\}$

Sets

- An *ordered pair* is a list (x, y)
  - *Order* matters: $(x, y) \neq (y, x)$ unless ... $x = y$
- The *Cartesian product* of two sets A, B
  - $\mathrm{A} \times \mathrm{B} = \{(a, b) | a \in \mathrm{A}, b \in \mathrm{B}\}$
- Example: The set of all points in the plane
- Cartesian products of three or more sets
- Cartesian *powers* of sets
  - $\mathrm{A}^n = \mathrm{A} \times \mathrm{A} \times \cdots \times \mathrm{A} = \{(a_1, a_2, \ldots, a_n) | a_1, a_2 \ldots, a_n \in \mathrm{A}\}$
- Some common examples:
  - $\mathbb{R}^2$
  - $\mathbb{R}^3$
  - $\mathbb{Z}^2$

Sets

- Set A is a *subset* of set B if
  - *Every* element of A is also an element of B
  - Written $A \subseteq B$
  - The negation is written $A \nsubseteq B$
    - When is this true?
  - For *any* set A, $\emptyset \subseteq A$

Sets

- The *union* of two sets A and B is $A \cup B = \{x | (x \in A) \vee (x \in B)\}$
- The *intersection* of A and B is $A \cap B = \{x | (x \in A) \wedge (x \in B)\}$
- The *set difference* of A and B is $A \setminus B = \{x | (x \in A) \wedge (x \notin B)\}$
  - $[1, 10] \cup [7, 12] = [1, 12]$
  - $[1, 10] \cap [7, 12] = [7, 10]$
  - $[1, 10] \setminus [7, 12] = [1, 7)$
  - $[1, 10] \setminus [5, 7] = [1, 5) \cup (7, 10]$

Sets

- *Universal set*
  - A set which contains all “currently interesting” sets as its subsets
  - Is not explicitly stated in many cases
  - If not explicitly given, we implicitly use a natural candidate
    - E.g: When considering primes, *usually* the universal set is $\mathbb{N}$
    - And not—say—$\mathbb{R}$, unless explicitly stated
  - Denoted U unless a named universal set is specified
- The *complement* of a set
  - $\bar{A} = U \setminus A$

Sets

- $A_1 \cup A_2 \cup \cdots \cup A_n = \{x \mid x \in A_i \text{ for at least one } 1 \leq i \leq n\}$
- $A_1 \cap A_2 \cap \cdots \cap A_n = \{x \mid x \in A_i \text{ for every } 1 \leq i \leq n\}$
- Notation:

$$\bigcup_{i \in [n]} A_i = \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$\bigcap_{i \in [n]} A_i = \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

Sets

When the list of sets is infinite:

$$\bigcup_{i\in\mathbb{N}} A_i = \bigcup_{i=1}^{\infty} A_i \quad = A_1 \cup A_2 \cup \cdots = \{x \,|\, x \in A_i \text{ for at least one } 1 \leq i\}$$

$$\bigcap_{i\in\mathbb{N}} A_i = \bigcap_{i=1}^{\infty} A_i \quad = A_1 \cap A_2 \cap \cdots = \{x \,|\, x \in A_i \text{ for every } 1 \leq i\}$$

Sets

- More generally, the set of indices can be *any* set
  - Not just $\{1, 2, \ldots, n\}$ or $\mathbb{N}$
- Let I be a set. Once we (somehow) associate a set A_α for each $\alpha \in I$,

$$\bigcup_{\alpha \in I} A_\alpha = \{x \mid x \in A_\alpha \text{ for at least one } \alpha \in I\}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{x \mid x \in A_\alpha \text{ for every } \alpha \in I\}$$

- Indexed collections of sets
  - I is the index set

Sets

- Why are indexed collections useful?
- Example: Stating the Principle of Inclusion and Exclusion

Theorem (Principle of Inclusion and Exclusion)

Let U be a universe and let $A_1, \ldots, A_n \subseteq U$. Then

$$|\bigcap_{i \in [n]} A_i| = \sum_{X \subseteq [n]} (-1)^{|X|} |\bigcap_{i \in X} \bar{A}_i|,$$

where

$$\bigcap_{i \in \emptyset} \bar{A}_i \triangleq U.$$

Relations

- Examples of mathematical relations?
- Formal way of definiting relations:
  - A *relation* on a set A is a subset $R \subseteq A \times A$
  - $(a, b) \in R$ is often written as aRb
  - $(a, b) \notin R$ is often written as $a\not{R}b$
- An example:
  - $A = \{1, 2, 3, 4\}$
  - $R = \{(2, 1), (3, 2), (3, 1), (4, 3), (4, 2), (4, 1)\} \subseteq A \times A$
  - What is the relation R?
  - What would be the relation $\not{R}$?
- *Every* mathematical relation on A is a subset of $A \times A$
  - Conversely, *every* subset of $A \times A$ is (*by definition*) a relation on A
  - Such a subset may not have a name, though.
- What does the subset corresponding to the relation $=$ on $\mathbb{R}$ look like?

Relations

Let R be a relation on a set A

- $x\mathrm{R}y$ is a *predicate*
  - Why?
  - We can use logical operators on such *relational expressions*
- R is *reflexive* if $\forall x \in \mathrm{A}\ \ x\mathrm{R}x$
  - Example?
  - Non-example?
- R is *symmetric* if $\forall x, y \in \mathrm{A}\ \ x\mathrm{R}y \implies y\mathrm{R}x$
  - Example?
  - Non-example?
- R is *transitive* if $\forall x, y, z \in \mathrm{A}\ \ (x\mathrm{R}y \wedge y\mathrm{R}z) \implies x\mathrm{R}z$
  - Example?
  - Non-example?

Relations

Let R be a relation on a set A

- R is an *equivalence* relation on A if
  - R is reflexive, symmetric, and transitive.
  - Examples?
  - Non-examples?
  - Equivalence relations are *very* special!
- Let R be an equivalence relation on set A, and let $a \in A$
  - $\{x \in A \mid xRa\}$ is the *equivalence class containing a*
  - Denoted $[a]$. Thus $[a] = \{x \in A \mid xRa\}$

Relations

Theorem

Let R be an equivalence relation on a set A, and let $a, b, \in A$. Then $[a] = [b] \iff aRb$.

Definition (Partition)

A *partition* of a set A is a collection $\mathcal{S}$ of subsets of the set A such that:

1. $\emptyset \notin \mathcal{S}$
2. $\bigcup_{X \in \mathcal{S}} X = \mathrm{A}$
3. $\forall X, Y \in \mathcal{S} \;\; (X \neq Y \implies X \cap Y = \emptyset)$

Theorem

Let R be an equivalence relation on a set A. Then the equivalence classes of R partition the set A.

Thank You!