Basic Mathematical Techniques for Computer Scientists

Functions

November 19, 2012

Recap

Recap

- Sets: Well-defined collections
  - Ways of expressing sets
  - Equality of sets
  - Cardinality of finite sets
  - The empty set

Recap

- Sets: Well-defined collections
- Intervals on the real line: $[], (], [), ()$

Recap

- Sets: Well-defined collections
- Intervals on the real line: $[], (], [), ()$
- Ordered pairs
  - Cartesian product of two or more sets
  - Cartesian powers of a set

Recap

- Sets: Well-defined collections
- Intervals on the real line: $[], (], [), ()$
- Ordered pairs
- Basic set operations
  - Subset, union, intersection, difference
  - Universal sets and the complement of a set
  - Indexed collections of sets

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- Intervals on the real line: $[], (], [), ()$
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- Relations: Subsets of cartesian products

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- Sets: Well-defined collections
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- Basic set operations
- Relations: Subsets of cartesian products
- Special kinds of relations
  - Reflexive
  - Symmetric
  - Transitive
  - Equivalence

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- Sets: Well-defined collections
- Intervals on the real line: $[], (], [), ()$
- Ordered pairs
- Basic set operations
- Relations: Subsets of cartesian products
- Special kinds of relations
- More on equivalence relations
  - Equivalence classes
  - Partitioning a set

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- Sets: Well-defined collections
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- Relations: Subsets of cartesian products
- Special kinds of relations
- Equivalence classes and partitions
- Two theorems about equivalence relations:
  - Two elements are related *iff* their equivalence classes coincide.

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- Two theorems about equivalence relations:
  - Two elements are related *iff* their equivalence classes coincide.
  - Equivalence classes partition the underlying set.

Recap

Questions?

- Sets: Well-defined collections
- Intervals on the real line: $[], (], [), ()$
- Ordered pairs
- Basic set operations
- Relations: Subsets of cartesian products
- Special kinds of relations
- Equivalence classes and partitions
- Two theorems about equivalence relations:
  - Two elements are related *iff* their equivalence classes coincide.
  - Equivalence classes partition the underlying set.

Functions

- A relation *between* sets A and B:
  - Is a subset of $A \times B$
  - $(a, b) \in R$ is often written as aRb
  - $(a, b) \notin R$ is often written as $a\not{R}b$
  - Examples?
- A *function* f from set A to set B:
  - Is denoted $f : A \rightarrow B$
  - Is a relation from A to B
  - $f \subseteq A \times B$
  - For each $a \in A$ there is *exactly one* $b \in B$ such that: $(a, b) \in f$
    - $(a, b) \in f$ is also written as: $f(a) = b$
  - Examples?
  - Non-examples?
- A function f from set A to Set B
  - Is *not necessarily* a short "rule" to convert from A to B ...
  - ... though many functions you come across will be/have such rules.

Functions

Some definitions

- A relation between sets A and B is a subset of $A \times B$
- A function f from A to B is a special relation from A to B
  - Each element of the set A "occurs" exactly once in f
- Given $f : A \to B$
  - The set A is the *domain* of function f
  - The set B is the *codomain* of function f
  - The set $\{f(x) \mid x \in A\}$ is the *range* of f
    - This is the same as $\{b \in B \mid (a, b) \in f\}$
    - Why two names, *codomain* and *range*?
    - Can these be different?
    - Examples?

Functions

Equality

- When are two functions $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{C} \to \mathrm{D}$ equal?
- These functions f, g are defined to be equal if
  - $A = C, B = D$
  - $\forall a \in \mathrm{A}\ f(a) = g(a)$.
  - Equal domains, equal codomains, and $f = g$ as sets.

Functions

Some more names

- Function $f : A \to B$ is *one-one* or *injective* if
  - $\forall x, y \in A \;\; x \neq y \implies f(x) \neq f(y)$
- Can there be another kind?
  - The other kind is called *many-one*
- Examples of injective functions?
- Non-examples?
- Function $f : A \to B$ is *onto* or *surjective* if
  - $\forall y \in B \, \exists x \in A \,|\, f(x) = y$
- Can there be another kind?
- Examples of surjective functions?
- Non-examples?

Functions

Some more names

- Function $f : A \to B$ is
  - *one-one* or *injective* if $\forall x, y \in A \;\; x \neq y \implies f(x) \neq f(y)$
  - *onto* or *surjective* if $\forall y \in B \, \exists x \in A \,|\, f(x) = y$
  - *bijective*—or *one-one and onto*—if it is *both* injective and surjective
- Example of a bijective function (*bijection*)?
- Non-example?
- These notions are widely used.

Functions

Some examples

- Function $f : \mathrm{A} \to \mathrm{B}$ is
  - *one-one* or *injective* if $\forall x, y \in \mathrm{A}\;\; x \neq y \implies f(x) \neq f(y)$
  - *onto* or *surjective* if $\forall y \in \mathrm{B}\, \exists x \in \mathrm{A} \,|\, f(x) = y$
  - *bijective* if it is *both* injective and surjective
- Let $f : (\mathbb{R} \setminus \{0\} \to \mathbb{R})$ be defined by $f(x) = 1 + \frac{1}{x}$
  - Is f injective?
  - Is f surjective?
- Let $f : (\mathbb{R} \to \mathbb{R})$ be defined by $f(x) = x^2$
  - Is f injective?
  - Is f surjective?

The Pigeonhole Principle

Theorem (The Pigeonhole Principle)

Let A and B be two finite sets, and let $f : A \to B$ be some function. Then

- *If $|A| > |B|$, then f is not injective.*
- *If $|A| < |B|$, then f is not surjective.*

- While this sounds quite innocent, it is mighty useful.
  - In fact, it is kind of awesome . . .

The Pigeonhole Principle

An example of its application

Definition

Two natural numbers are said to be *coprime* or *mutually prime* if they have exactly one common divisor, which is 1.

- The numbers themselves do *not* have to be prime.
- Examples?

Fact

Let A be ***any*** *set of eleven natural numbers chosen from* $\{1, 2, \ldots, 20\}$. *Then the set A contains at least two numbers which are coprime.*

Fact

Let A be any set of $(n+1)$ *natural numbers chosen from* $\{1, 2, \ldots, 2n\}$. *Then the set A contains at least two numbers which are coprime.*

The Pigeonhole Principle

Another example

Fact

*Let A be **any** set of ten natural numbers chosen from $\{1, 2, \ldots, 100\}$. Then there are two subsets X and Y of A such that*

- *$X \neq Y$, and,*
- *The sums of the numbers in X and in Y are **equal**.*

Cardinalities of sets

- When do two sets A and B have the same cardinality?
- The naïve notion does not suffice when
  - It is too difficult to count their elements
  - More importantly: when the sets are not finite.
  - We need a different notion ...

Definition (Cardinalities of sets)

Two sets A and B are said to have the **same cardinality** if there exists a bijection $f : A \to B$. In this case we write $|A| = |B|$.
If no such bijective function exists, then the sets have **unequal cardinalities**, and we write $|A| \neq |B|$.

- The definition applies to finite sets *and* to infinite sets.
  - Example: $|\mathbb{N}| = |\mathbb{Z}|$
  - Example: $|\mathbb{N}| \neq |\mathbb{R}|$

Countable sets, uncountable sets

Definition

The cardinality of the empty set is zero. Let A be a nonempty set.

- The set A is said to be **finite** if there is positive integer n such that there is a bijection from A to the set $\{1, \ldots, n\}$. The set A is said to be **infinite** otherwise.
- The set A is said to be **countable** if there is a bijection from A to some subset of $\mathbb{N}$.
- The set A is **countably infinite** if there is a bijection from A to $\mathbb{N}$.
- The set A is **uncountable** (or *uncountably infinite*) if A is infinite and there exists no bijection from A to $\mathbb{N}$.

Thank You!