Basic Mathematical Techniques for Computer Scientists

Counting

November 26, 2012

Recap

Recap

- Functions: A special kind of relation
  - More abstract than "rules"
  - Domain, co-domain, range
  - Equality of functions

Recap

- Functions: A special kind of relation
- Special kinds of functions:
  - One-one (injective)
  - Many-one
  - Onto (surjective)
  - One-one and onto (bijective)

Recap

- Functions: A special kind of relation
- One-one, many-one, onto, and bijective functions
- The Pigeonhole Principle: For finite sets A and B and $f : A \to B$,
  - $|A| > |B| \implies f$ is not one-one
  - $|A| < |B| \implies f$ is not onto

Recap

- Functions: A special kind of relation
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- Comparing the cardinalities of two sets
  - $|A| = |B| \iff$ there is a bijection from A to B.

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  - $|A| = |B| \iff$ there is a bijection from A to B.
- Finite, countable, and uncountably infinite sets

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Questions?

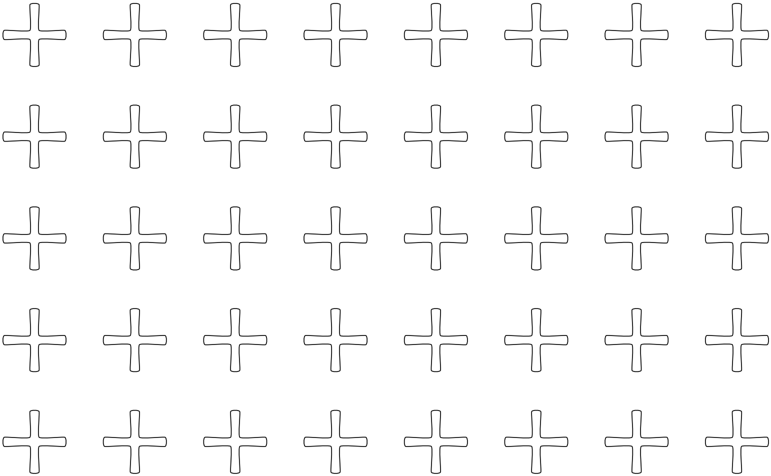
Counting

Counting

- Finding the *how many* things are there
  - In a set, or some other kind of object
- Why is this important in CS?
  - Used all over CS. E.g:
    - The analysis of algorithms
    - Discrete probability
- How do we count the number of objects in (say) a set?
  - One way: Tick off the objects against $1, 2, \ldots$
  - Are there other ways?

An example

How many crosses?



An example

How many crosses?

- How did we count the crosses?
  - We did *not* tick them off $1, 2, \ldots$
    - This is the stupid way to count these
  - Bijection from the set of crosses to $\{1, 2, \ldots, 5\} \times \{1, 2, \ldots, 8\}$
  - The rule for the cardinality of a Cartesian product
- A general idea for counting large and/or complicated sets:
  - Come up with a bijection from the set we want to count
    - To some set which we *know* how to count
  - We will see many specific instances of this idea

Two basic counting rules

The Sum Rule

Theorem (The Sum Rule)

If $A_1, A_2 \ldots, A_n$ are finite ***disjoint*** *sets, then*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

- How can we *prove* this?
- Does this hold for sets which *intersect*?

Two basic counting rules

The Product Rule

Theorem (The Product Rule)

If $A_1, A_2 \ldots, A_n$ are finite sets, then

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_n|.$$

- How can we *prove* this?
- Does this hold for sets which *intersect*?

Using the basic counting rules

An example

- A website's restrictions on your password:
  - From six to eight symbols long
  - The first symbol must be a letter from the English alphabet
  - The rest must be letters or digits
  - *How many* different passwords are possible?

Using the basic counting rules

Another example

- How many subsets does an n-element set have?

Some terminology

Definition (List/String/Sequence)

Let $A = \{a_1, a_2 \ldots, a_n\}$ be a finite set, and let $k \in \mathbb{N}$. A *list of k elements drawn from the set A* is an element of A^k. Such a list is also called a *sequence* or *string over* the set A. The *length* of such a list (string/sequence) is k.

- Sometimes written without parens and commas.
- When using "string", the set A is called the *alphabet*.

Example

$(1, 1, 1, 1)$ is a list of 4 elements drawn from the set $\{0, 1\}$. It is also:

- written as 1111 when there is no scope for confusion;
- a sequence of length 4 over $\{0, 1\}$, and,
- a string of length 4 over the alphabet $\{0, 1\}$.

Strings over an alphabet

An example

- Consider the alphabet A $= \{a, b, c, d, e, f, g\}$.
  - Example of a string of length 4 over A?
  - How many strings of length four over A?
  - How many strings of length four over A, if no letter should repeat?

Generalized product rule

Theorem (Generalized Product Rule)

Let $k \in \mathbb{N}; k \geq 2$, and let S be a set of sequences of length k over some set A. If there are:

- *n_1 possible ways of choosing an element of A as the first element in a sequence of S;*
- *given any choice of the first element, there are n_2 possible ways of choosing the second element in a sequence of S;*
- *given any choice of the first two elements, there are n_3 possible ways of choosing the second element in a sequence of S;*
- *and so on up till the kth element, then*

$$|S| = n_1 \cdot n_2 \cdot \cdots \cdot n_k.$$

- How can we *prove* this?

Strings over an alphabet

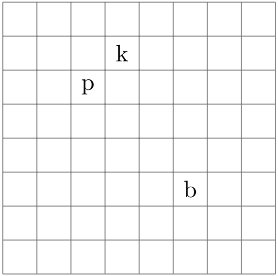
The example, continued

- Consider the alphabet $A = \{a, b, c, d, e, f, g\}$.
  - How many strings of length four over A which contain the letter a, and has no repeating letter?
  - How many strings of length four over A which contain the letter a?

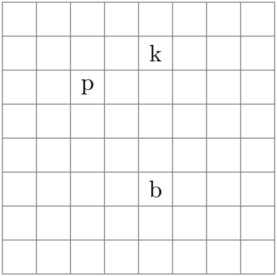
Example

Three chessmen on a board

- In how many ways can we place a pawn (p), a knight (k), and a bishop (b) on an (8×8) chessboard such that no two of these share a row or a column?



(a) Valid placement.

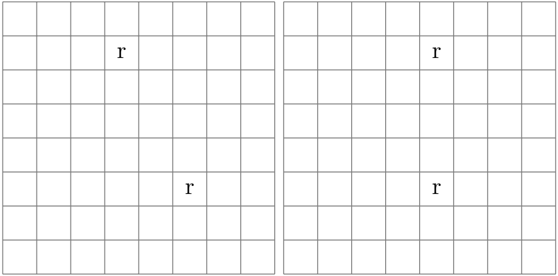


(b) Invalid placement.

Example

Two rooks of the same colour

- In how many ways can we place two black rooks on an (8×8) chessboard such that they don't share a row or a column?



(a) Valid placement. (b) Invalid placement.

The Quotient Rule

Definition (k-to-1 functions)

For any positive integer k, a function $f : \mathrm{A} \to \mathrm{B}$ is said to be **k-to-1** if, for each $y \in \mathrm{B}$, $(\exists x \in \mathrm{A} \mid f(x) = y) \implies |\{x \in \mathrm{A} \mid f(x) = y\}| = k$.

Theorem (Quotient Rule)

Let $k \in \mathbb{N}$. If a function $f : A \to B$ is k-to-1 and onto, then $|A| = k \cdot |B|$.

Permutations

- A *permutation* of a set A is
  - A list of *all* the elements in the set A
  - Of length $|A|$: Each element appears *exactly once*
- Which of the following are permutations of $\{1, 2, 3, 4, 5\}$?
  - $(1, 2, 3, 4, 5)$
  - $(1, 4, 1, 2, 3)$
  - $(5, 4, 3, 2, 1)$
  - $(1, 4, 5, 2, 3)$
  - $(1, 4, 5, 2, 3, 5)$
- How many permutations does an n-element set have?

Permutations

The factorial function

Definition (The factorial function)

For $n \in \mathbb{N}$, the **factorial** of n, denoted $n!$, is defined to be the number of permutations of a set with n elements.

- From the definition, we get:
  - $0! = 1$
  - $1! = 1$

Theorem

For $n \in \mathbb{N}, n > 1$*,* $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1.$

Example

Seating people

- In how many different ways can we seat n people in a row?
- In how many different ways can we seat n people around a round table, if
  - Absolute positions matter
    - There is a special seat
  - Absolute positions do not matter
    - No special seat
    - Only relative positions matter

Thank You!