Basic Mathematical Techniques for Computer Scientists

Counting, Part II

December 3, 2012

Recap

- Counting: Finding *how many* things there are
  - Very important in CS
  - Simplest way: tick off things against $1, 2, \ldots$
    - Not practical in most interesting cases
  - Other ways to count
  - General idea: Bijection from the known to the unknown
    - Or in the other direction

Recap

- Counting: Finding *how many* things there are
- Use bijections to make counting easier
- Two basic counting rules:
  - The Sum Rule
  - The Product Rule

Recap

- Counting: Finding *how many* things there are
- Use bijections to make counting easier
- Two basic counting rules.
- Some terminology:
  - List, sequence, string
  - The length of a list/sequence/string
  - Alphabet

Recap

- Counting: Finding *how many* things there are
- Use bijections to make counting easier
- Two basic counting rules
- List, sequence, string, alphabet, length
- Generalized Product Rule
  - Very useful to count the number of sequences with extra restrictions

Recap

- Counting: Finding *how many* things there are
- Use bijections to make counting easier
- Two basic counting rules
- List, sequence, string, alphabet, length
- Generalized Product Rule
- The Quotient Rule
  - Used to get the true count from an "overcount"

Recap

- Counting: Finding *how many* things there are
- Use bijections to make counting easier
- Two basic counting rules
- List, sequence, string, alphabet, length
- Generalized Product Rule
- The Quotient Rule
- Permutations and the factorial function
  - Permutation: List of all the elements in a set, exactly once
  - Factorial of $n \in \mathbb{N}$: The number of permutations of a set of size n.

Questions?

Counting lists made up from a set

- Let A be a finite set with n elements
- How many lists of length k, of elements of A?
  - If we allow elements to repeat in a list: n^k
    - This makes sense only when $k \geq 0$
  - If we do not allow elements to repeat in any list: $n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)$
    - This is equal to $\frac{n!}{(n-k)!}$
    - This makes sense only when $0 \leq k \leq n$

Counting lists made up from a set

Two theorems

Theorem (Number of lists, no restrictions)

The number of lists of length $k \geq 0$ of elements chosen from a set of size n is

$$n^k.$$

Theorem (Number of lists, no repetitions)

The number of lists of length $0 \leq k \leq n$ of elements chosen from a set of size n, when no element repeats in a list, is

$$\frac{n!}{(n-k)!}.$$

A digression (important!)

- You should *not* memorize the formulas in these theorems
  - Also true for the other formulas in this course
  - And in most other courses
- If you spend effort in memorizing these formulas
  - You will most probably not remember *why* they are true
    - You will have forgotten the mathematics soon
    - You will only be able to say "It's a rule"
    - A failure for you (and your teacher)
  - You are likely to make silly mistakes when applying them
    - It is easy to misplace a bracket, for example

A digression (important!)

- You should *not* memorize the formulas in these theorems
  - Also true for the other formulas in this course
  - And in most other courses
- Instead, each time you do an exercise,
  - Recall the *argument* which gets you the rule
    - Take the help of the slides/Wikipedia/whatever
  - If you do this,
    - You can *happily* forget the rule
    - Because you *know* that when you need it, . . .
    - . . . you can derive it yourself!
    - You will *internalize* the mathematics
    - You will end up *smarter* than Joe Learnbyrote

Counting the subsets of a set

- Goal: Find the number of subsets of size k, of a set A of size $n \in \mathbb{N}$.
  - These are called the k-subsets of A
  - In general, a set of size n is sometimes called an n-set
- Use what we already know:
  - The number of *lists* of size k, with or without repetition
  - Various lemmas/theorems which we saw so far
- Question: How do we go from *lists* to *subsets*?
  - How are these two different?
  - How do we make use of this?

Counting the subsets of a set

Theorem (The number of subsets of a certain size)

Let A be a set of size $n \in \mathbb{N}$, and let $k \in \mathbb{Z}$.

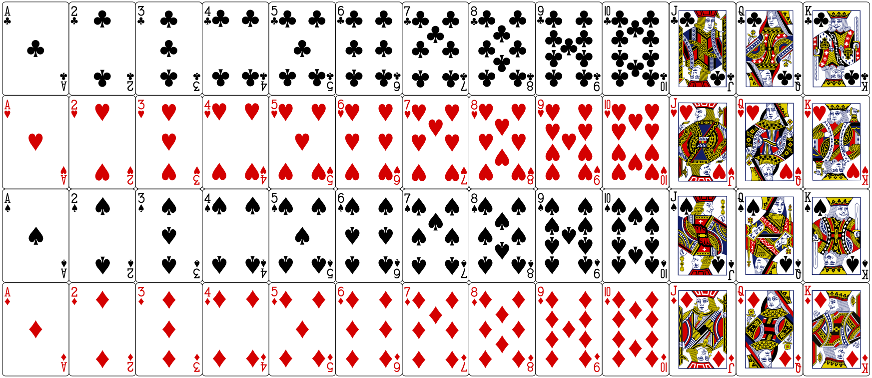
- *If $k < 0$ or $k > n$, then the number of k-subsets of A is zero.*
- *Otherwise, the number of k-subsets of A is $\frac{n!}{k!(n-k)!}$.*

- Notation: We use $\binom{n}{k}$ to denote the number of k-subsets of a set with n elements.
- So the theorem says:
  - If $k < 0$ or $k > n$, then $\binom{n}{k} = 0$
  - Otherwise, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Playing Cards

A deck of 52 cards

- 4 *suits*, 13 cards in each suit



Vectorized Playing Cards 1.3- `http://code.google.com/p/vectorized-playing-cards/`

Playing Cards

A deck of 52 cards

- 4 *suits*, 13 cards in each suit
- A *hand* is a set of cards dealt from the deck
  - How many different 6-card hands are there?
  - How many 6-card hands, with 2 spades and 4 clubs?
  - How many 7-card hands, with 2 spades, 2 clubs, and 3 hearts?

A couple of useful facts

About the numbers $\binom{n}{k}$

- We *defined* $\binom{n}{k}$ as the number of k-subsets of any n-set

A couple of useful facts

About the numbers $\binom{n}{k}$

- We *defined* $\binom{n}{k}$ as the number of k-subsets of any n-set
- Claim:

$$\binom{n}{n} = 1$$

  - Proof?
- Claim:

$$\binom{n}{0} = 1$$

  - Proof?
- Claim:

$$\binom{n}{k} = \binom{n}{n-k}$$

  - Proof?
- Claim:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

  - Proof?

Example

Counting bits

- What is a bit?
- What is an n-bit vector?
- How many n-bit vectors have exactly k ones in them?

Theorem

For any $n \in \mathbb{N}$ and $k \in \mathbb{N}, 0 \leq k \leq n$, the number of n-bit vectors with exactly k ones is $\binom{n}{k}$.

Monomials

Definition (Monomials)

A monomial is a product of non-negative integral constant powers of variables.

Example

- Some monomials: $x, y, xy, a^2bc, a^2b^3c^5x^{10}y$
- Some non-monomials: $x + y, a(b + x^2), a^3 + 3bx^2 - cx - yz, 7abc - 10xy$
- Some other non-monomials: $\sin x, \ln x, x\sqrt{xy}, x^x$

- The second list of non-monomials are all examples of *polynomials*—these are expressions formed by adding or subtracting two or more monomials.
- The prefix "*poly*" means "many".

Some monomials

Spot the pattern

Polynomial	Monomials	Count
$(x+y)$	$\{x, y\}$	2
$(x+y)^2$	$\{x^2, xy, xy, y^2\}$	4
$(x+y)^3$	$\{x^3, x^2y, x^2y, x^2y, xy^2, xy^2, xy^2, y^3\}$	8
$(x+y)^4$	$\{x^4, x^3y, x^3y, x^3y, x^2y^2, x^2y^2, x^2y^2, xy^3, x^3y, x^2y^2, x^2y^2, x^2y^2, xy^3, xy^3, xy^3, y^4\}$	16

- Given a list L_{100} of all the monomials the expansion of $(x+y)^{100}$
  - How will you create the list L_{101} of the monomials of $(x+y)^{101}$?
- What pattern do the numbers in the rightmost column follow?

Spot the pattern

Relate it to something which we know

- Do we know something else which follows the same pattern of numbers?
- How do we “connect” the two?
  - “Why” this pattern among counts of monomials?
  - How do we map what we know to what we see?
  - What is a *bijection* from one to another?

Binomials

Definition

Definition (Binomials)

A *binomial* is a polynomial with exactly two terms—it is the sum (or difference) of two monomials.

- Examples: $x + y, x^7a^3b - abxy$

Theorem (The Binomial Theorem)

*For any $n \in \mathbb{N}$, each **monomial** in the expansion of $(x+y)^n$ is of the form $x^i y^{n-i}$; $0 \le i \le n$.*
For each $k \in \mathbb{N}, 0 \le k \le n$, the monomial $x^k y^{n-k}$ appears exactly $\binom{n}{k}$ times in this expansion.
*So the **coefficient** of $x^k y^{n-k}$ in $(x+y)^n$ is $\binom{n}{k}$.*
Thus,

$$(x+y)^n = \binom{n}{n} x^n y^0 + \binom{n}{n-1} x^{n-1} y + \cdots + \binom{n}{n-i} x^{n-i} y^i + \cdots + \binom{n}{0} x^0 y^n.$$

Using the equations we saw before, this is usually written:

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{i} x^{n-i} y^i + \cdots + y^n.$$

Thank You!