Braess's paradox

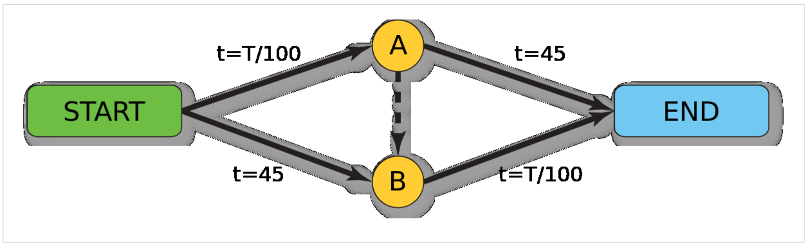
Braess's paradox, credited to the German mathematician Dietrich Braess (de), states that adding extra capacity to a network when the moving entities selfishly choose their route, can in some cases reduce overall performance. This is because the Nash equilibrium of such a system is not necessarily optimal.

The paradox is stated as follows: "For each point of a road network, let there be given the number of cars starting from it, and the destination of the cars. Under these conditions one wishes to estimate the distribution of traffic flow. Whether one street is preferable to another depends not only on the quality of the road, but also on the density of the flow. If every driver takes the path that looks most favorable to him, the resultant running times need not be minimal. Furthermore, it is indicated by an example that an extension of the road network may cause a redistribution of the traffic that results in longer individual running times."

The reason for this is that in a Nash equilibrium, drivers have no incentive to change their routes. If the system is not in a Nash equilibrium, selfish drivers must be able to improve their respective travel times by changing the routes they take. In the case of Braess's paradox, drivers will continue to switch until they reach Nash equilibrium, despite the reduction in overall performance.

If the latency functions are linear then adding an edge can never make total travel time at equilibrium worse by a factor of more than 4/3.[1]

Example



Consider a road network as shown in the adjacent diagram, on which 4000 drivers wish to travel from point Start to End. The travel time in minutes on the Start-A road is the number of travelers (T) divided by 100, and on Start-B is a constant 45 minutes (likewise with the roads across from them). If the dashed road does not exist (so the traffic network has 4 roads in total), the time needed to drive Start-A-End route with A drivers would be $\frac{A}{100}+45$. And the time needed to drive the Start-B-End route with B drivers would be $\frac{B}{100}+45$. If either route were shorter, it would not be a Nash equilibrium: a rational driver would switch routes from the longer route to the shorter route. As there are 4000 drivers, the fact that $A+B=4000$ can be used to derive the fact that $A=B=2000$ when the system is at equilibrium. Therefore, each route takes $\frac{2000}{100}+45=65$ minutes.

Now suppose the dashed line is a road with an extremely short travel time of approximately 0 minutes. In this situation, all drivers will choose the Start-A route rather than the Start-B route, because Start-A will only take $\frac{T}{100}=\frac{4000}{100}=40$ minutes at its worst, whereas Start-B is guaranteed to take 45 minutes. Once at point A, every rational driver will elect to take the "free" road to B and from there continue to End, because once again A-End is guaranteed to take 45 minutes while A-B-End will take at most $0+\frac{4000}{100}=40$ minutes. Each driver's travel time is $\frac{4000}{100}+\frac{4000}{100}=80$ minutes, an increase from the 65 minutes required when the fast A-B road did not exist. No driver has an incentive to switch, as the two original routes (Start-A-End and Start-B-End) are both now 85 minutes. If every driver were to agree not to use the A-B path, every driver would benefit by reducing their travel time by 15 minutes. However, because any single driver will always benefit by taking the A-B path, the socially optimal distribution is not stable and so Braess's paradox occurs.

Existence of an equilibrium

Let $L_e(x)$ be the formula for the cost of x people driving along edge e. If a traffic graph has linear edges (those of the form $L_e(x) = a_e x + b_e > 0$ where a_e and b_e are constants) then an equilibrium will always exist.
Suppose we have a linear traffic graph with e_x people driving along edge e. Let the energy of e, $E(e)$ be

$$\sum_{i=1}^{e_x} L_e(i) = L_e(1) + L_e(2) + ... + L_e(e_x)$$

(If $x_e = 0$ let $E(e) = 0$). Let the total energy of the traffic graph be the sum of the energies of every edge in the graph.
Suppose that the distribution for the traffic graph is not an equilibrium. There must be at least one driver who can switch their route and improve total travel time. Suppose their original route is $x_0, x_1, ...x_n$ while their new route is $y_0, y_1, ...y_m$. Let E be total energy of the traffic graph, and consider what happens when the route $x_0, x_1, ...x_n$ is removed. The energy of each edge x_i will be reduced by $L_e(e_x)$ and so the E will be reduced by $\sum_{i=0}^{n} L_e(e_x)$. Note that this is simply the total travel time needed to take the original route. If we then add the new route, $y_0, y_1, ...y_m$, E will be increased by the total travel time needed to take the new route. Because the new route is shorter than the original route, E must decrease. If we repeat this process, E will continue to decrease. As E must remain positive, eventually an equilibrium must occur.

Finding an equilibrium

The above proof outlines a procedure known as Best Response Dynamics, which finds an equilibrium for a linear traffic graph and terminates in a finite number of steps. The algorithm is termed "best response" because at each step of the algorithm, if the graph is not at equilibrium then some driver has a best response to the strategies of all other drivers, and switches to that response.

Pseudocode for Best Response Dynamics:

```
Let P be some traffic pattern.
while P is not at equilibrium:
  compute the potential energy e of P
  for each driver d in P:
    for each alternate path p available to d:
       compute the potential energy n of the pattern when d takes path p
       if n < e:
         modify P so that d takes path p
         continue the topmost while
```

At each step, if some particular driver could do better by taking an alternate path (a "best response"), doing so strictly decreases the energy of the graph. If no driver has a best response, the graph is at equilibrium. Since the energy of the graph strictly decreases with each step, the Best Response Dynamics algorithm must eventually halt.

How far from optimal is traffic at equilibrium

At worst, traffic in equilibrium is twice as bad as socially optimal[1]

Proof

S_j = starting point for car j

T_j = target for car j

Strategies for car j are possible paths from S_j to T_j

Each edge e has a travel function $L_e(x) = a_e x + b_e$ for some $a_e, b_e \geq 0$

Energy on edge e with x drivers:

$$E(e) = L_e(1) + L_e(2) + \cdots + L_e(x)$$

Total time spent by all drivers on that edge:

$$T(e) = xL_e(x)$$ ((where there are x terms))

E(e) is less than or equal to T(e) and

$$\begin{aligned} L_e(1) + \cdots + L_e(x) &= a_e(1 + 2 + \cdots + x) + b_e x \\ &= a_e \tfrac{x(x+1)}{2} + b_e x \\ &= x(a_e \tfrac{x+1}{2} + b_e) \\ &\geq \tfrac{1}{2} x(a_e x + b_e) \\ &= \tfrac{1}{2} T(e) \end{aligned}$$

Resulting Inequality

$$\tfrac{1}{2} T(e) \leq E(e) \leq T(e)$$

If Z is a traffic pattern:

$$\tfrac{1}{2} SocialCost(Z) \leq E(Z) \leq SocialCost(Z)$$

If we start from a socially optimal traffic pattern Z and end in an equilibrium pattern Z':

$$\begin{aligned} SocialCost(Z') &\leq 2E(Z') \\ &\leq 2E(Z) \\ &\leq 2SocialCost(Z) \end{aligned}$$

Thus we can see that worst is twice as bad as optimal.

How rare is Braess's paradox?

In 1983 Steinberg and Zangwill provided, under reasonable assumptions, necessary and sufficient conditions for Braess's paradox to occur in a general transportation network when a new route is added. (Note that their result applies to the addition of *any* new route — not just to the case of adding a single link.) As a corollary, they obtain that Braess's paradox is about as likely to occur as not occur; their result applies to random rather than planned networks and additions.

In Seoul, South Korea, a speeding-up in traffic around the city was seen when a motorway was removed as part of the Cheonggyecheon restoration project.[2] In Stuttgart, Germany after investments into the road network in 1969, the traffic situation did not improve until a section of newly-built road was closed for traffic again.[3] In 1990 the closing of 42nd street in New York City reduced the amount of congestion in the area.[4] In 2008 Youn, Gastner and Jeong demonstrated specific routes in Boston, New York City and London where this might actually occur and pointed out roads that could be closed to reduce predicted travel times.[5]

In 2012, scientists at the Max Planck Institute for Dynamics and Self-Organization demonstrated through computational modeling the potential for this phenomenon to occur in power transmission networks where power generation is decentralized.[6]

In 2012, an international team of researchers from Institut Néel (CNRS, France), INP (France), IEMN (CNRS, France) and UCL (Belgium) published in Physical Review Letters[7] a paper showing that Braess paradox may occur in mesoscopic electron systems. In particular, they showed that adding a path for electrons in a nanoscopic network paradoxically reduced its conductance. This was showed both by theroretical simulations and experiments at low temperature using as scanning gate microscopy.

Dynamics analysis of Braess's paradox

In 2013, Dal Forno and Merlone [8] interpret Braess paradox as a dynamical ternary choice problem. The analysis shows how the new path changes the problem. Before the new path is available the dynamics is the same as in binary choices with externalities, but the new path transforms it in a ternary choice problem. The addiction of an extra resource enriches the complexity of the dynamics. In fact, in this case, there can even be coexistence of cycles. This way, the implication of the paradox on the dynamics can be seen from both a geometrical and analytical perspective.

References

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Further reading

- D. Braess, Über ein Paradoxon aus der Verkehrsplanung. *Unternehmensforschung* 12, 258–268 (1969) (http://homepage.ruhr-uni-bochum.de/Dietrich.Braess/paradox.pdf) (http://homepage.rub.de/Dietrich.Braess/Paradox-BNW.pdf)
- Katharina Belaga-Werbitzky: „Das Paradoxon von Braess in erweiterten Wheatstone-Netzen mit M/M/1-Bedienern“ ISBN 3-89959-123-2
- Translation of the Braess 1968 article from German to English appears as the article "On a paradox of traffic planning," by D. Braess, A. Nagurney, and T. Wakolbinger in the journal Transportation Science, volume 39, 2005, pp. 446–450. More information (http://supernet.som.umass.edu/cfoto/braess-visit/braessvisit.html)
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External links

- Software Testing Paradoxes (http://msdn.microsoft.com/msdnmag/issues/05/12/TestRun/default.aspx)
- Braess's homepage (http://homepage.ruhr-uni-bochum.de/Dietrich.Braess/#paradox)
- Characterizing Braess's Paradox for Traffic Networks (http://tigger.uic.edu/~hagstrom/Research/Braess/)
- The Road Network Paradox (http://www.davros.org/science/roadparadox.html)
- An article on New York Times (http://www.nytimes.com/1990/12/25/health/what-if-they-closed-42d-street-and-nobody-noticed.html)

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