max planck institut informatik

SAARLAND UNIVERSITY
COMPUTER SCIENCE

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Exercises for Randomized Methods in Computer Science

http://www.mpi-inf.mpg.de/departments/d1/teaching/ws12/rmcs/

Assignment 2 Due: Wednesday, October 31, 2012

Exercise 0 (Reminder of basics) Make yourself familiar with the notions of a *Bernoulli trial* (or Bernoulli random variable), the *binomial distribution* and the *geometric distribution* (taking values in the positive integers). That is: learn their definitions (probability mass functions)! Where do they come from and how are they related? What are their expectations? Use any textbook or Wikipedia as source.

a) Prove from the probability mass function that the sum of the probabilities of all outcomes is one.

b) Prove that the expectations are as claimed.

c) A monkey types on a 26-letter keyboard that has lower-case letters only. Each letter is chosen independently and uniformly at random. If the monkey types one million letters, what is the expected number of times the sequence "proof" appears? [From the Mitzenmacher/Upfal book]

Exercise 1 (Coupon collector, expected time. Difficulty: elementary)

Let us return to the coupon collector problem from the previous assignment. We now aim at determining the expected number of rounds it takes to get all n coupon types. Denote by X the total number of rounds needed, by X_i, $i = 1, \dots, n$ the number of rounds in which you have $i-1$ different coupons (at the beginning of the round). Then $X = \sum_{i=1}^{n} X_i$. Use the above to determine $E(X_i)$, and from this, $E(X)$. Do some minimal research on the harmonic number to obtain that $n \ln(n) \le E(X) \le n \ln(n+1)$.

Exercise 2 (Cuts in graphs. First moment method. Difficulty: elementary) Let $G = (V, E)$ be a graph. For a subset $S \subseteq V$ of vertices, the cut induced by S is the set $\delta(S) := \{e \in E \mid |e \cap S| = 1\}$ of edges having one vertex in S and the other in $V \setminus S$. Give a very simple randomized algorithm that finds an S inducing a cut of expected size $\frac{1}{2}|E|$. Proof!

Think a little more and improve this result by something like a $(1 + \frac{1}{2n})$ factor.

Exercise 3 (Coloring hypergraphs. First moment method. Difficulty: (a) elementary, (b) moderate, (c) immediate fame) Let $E_1, \ldots, E_m$ be a collection of m sets each of cardinality $n \in \mathbb{N}$. Let $V := \bigcup_{i \in [m]} E_i$. The pair $H = (V, \{E_1, \ldots, E_m\})$ is called n-uniform hypergraph. We say that H is 2-colorable if its vertex set V can be colored with two colors such that none of the edges E_i is monochromatic. Formally, if there is an $f : V \to \{1,2\}$ such that for all $i \in [m]$ there are $u, v \in E_i$ such that $f(u) = 1$ and $f(v) = 2$.

a) Proof that if $m < 2^{n-1}$, then H is 2-colorable. Note: Some people might call this an application of the first moment method. Can you see why?
b) Proof that for $m = \Omega(n^2 2^n)$, there is an n-uniform hypergraph having m edges that is not 2-colorable. Hint: Again use the probabilistic method.
c) Writing $m(n) := \max\{m \in \mathbb{N} \mid \text{every } n\text{-uniform hypergraph having } m \text{ edges is 2-colorable}\}$, the above shows $m(n) \geq 2^{n-1}$ and $m = O(n^2 2^n)$. A better lower bound of $m(n) = \Omega(2^n \sqrt{n/\log n})$ is known. Task: Improve either bound.

Exercise 4 (Games. Difficulty: easy to moderate) Consider the following game, which could represent the up-or-out principle common in some business sectors. We have a board of n ranks numbered from n (the lowest) to 1 (the highest). At the start of the game, on each rank there may be some tokens. Each round of the game consists of the following actions:

1. The player called "head of team" proposes a subset of the tokens to be promoted.
2. The player called "human resources manager" accepts or rejects this proposal.
3. If he accepts, the proposed tokens move up one rank and the other tokens are removed from the board. If he rejects, the proposed tokens are removed and the others move up one rank. If a token on rank 1 becomes promoted, it goes to an exceptional rank 0 and stays there for ever.

The game ends if a token reaches rank 0 (in this case head of team wins), or if all tokens are removed (then human resources manager wins).

Since this is a perfect-information game without any randomness involved, for each possible initial position of the game there is a uniquely determined winner of the game (assuming that both players play clever). Determining this winner is the task in this problem.

Hints: I suggest to analyze a simple randomized strategy for human resource manager. This gives a first criterion for what is a winning position for human resources manager. Then show that this criterion actually detects all winning positions for human resources manager, that is, give a strategy for head of team ensuring him a win when the criterion is not fulfilled. There will be no randomness in this last part. Note: Describe a position of the game (whether initial or not) by the number x_i of tokens on each rank i).

Exercise 5 (Mastermind with two colors. Difficulty: Not yet really difficult, but needs some thought of what has to be done and then some thought of how to do this) Use the exercise from the last assignment to show that there is a set of $O(n/\log n)$ guesses in the Mastermind game with two colors and n positions such that no matter what the secret code is, these guesses together with the answers unique reveal the secret code. If you can't solve this, try the following weaker question, which needs (apart from one) the same arguments. Prove that if I fix a secret code, then you cast $O(n/\log n)$ random guesses (which I truthfully answer), then with high probability this information is sufficient to determine the secret code. Prove also that the stronger statement is asymptotically optimal.