Prof. Dr. Benjamin Doerr **Winter 2012/13**

Exercises for Randomized Methods in Computer Science

http://www.mpi-inf.mpg.de/departments/d1/teaching/ws12/rmcs/

Assignment 4 Due: Wednesday, November 14, 2012

Exercise 1 In the asynchronous rumor spreading models, each node has a Poisson clock ticking with rate $\lambda = 1$. If the clock ticks, the node does something, e.g., in the push model the node calls a random neighbor, if it is informed.

a) [very elementary] Read the English Wikipedia page for "Poisson process" up to Section 2.1 plus Section 4. Read as well the page for "exponential distribution" up to the statement on the mean (expectation) in Section 2.1 plus Section 2.2.

b) [elementary] Prove that k independent Poisson clocks with identical rates λ together tick like one clock with rate $k\lambda$. Hint: By the memoryless property, all we need to show is that the waiting time T for the next tick follows an exponential distribution X with rate $k\lambda$. To show that these two distributions T and X are identical, show that $\Pr(T \geq \alpha) = \Pr(X \geq \alpha)$.

c) [easy, can be done independently from (a) and (b)] The above imply the following observation: If at some time t exactly k vertices are informed, then the expected waiting time for the next tick of a clock of an informed vertex is $1/k$. Use this to give an exact formula for the expected time the asynchronous push rumor spreading process takes to inform all vertices. Simplify your formula to show that this number is $2H_{n-1}$, where H_n is the n-th Harmonic number (this last step might might be a little tricky).

Exercise 2 Prove that synchronized randomized rumor spreading in the push-model with probability at least $1-1/n$ informs the complete graph on n vertices in $O(\log n)$ rounds. I suggest to do this in the following three steps, but feel free to use other approaches. The steps can be solved independently. I use the expression “high probability” below for anything that is sufficiently likely to make the whole proof work out. There is lots of room, don’t worry.

a) Prove that with high probability in the first $64\log n$ rounds the initially informed vertex calls at least $16\log n$ different vertices.

b) Prove that if a round begins with at least $16\log n$ vertices informed, but at most $n/6$, then with high probability at the end of the round at least 50% more are informed. I suggest to give a lower bound for the expected number of newly informed vertices and then argue, that with high probability you don’t end up much worse. You might see problems with the independence of random variables. Try to re-interpret your random experiment suitably. Yes, this does not feel like the most elegant solution, we will see better methods in the lecture.

c) Prove that if at some time at least $n/6$ vertices are informed, then with high probability after $18e\ln(n)$ rounds, all are informed.

d) Put everything together.

Exercise 3 Consider the following game. I think of a number between 1 and n and you have to find it using yes/no questions only. By find I mean that you are absolutely sure that you know the number, or in other words, that I don’t have a way to cheat and change the number I thought of (without creating a conflict with my previous answers).

a) Show that you need at least $\log_2(n)$ questions.

b) Consider now the game where I am allowed to lie once. That is, to one of your questions I may give the wrong answer. Show that you now need at least $\log_2 n + \Theta(\log\log n)$ questions. Note: This is, by the way, the right order of magnitude, which interestingly can also be proven by a not too difficult randomized argument.