Prof. Dr. Benjamin Doerr **Winter 2012/13**

Exercises for Randomized Methods in Computer Science

http://www.mpi-inf.mpg.de/departments/d1/teaching/ws12/rmcs/

Assignment 5 Due: Wednesday, November 21, 2012

Exercise 0 Read the first section of Spencer's paper "Randomization, Derandomization and Antirandomization—Three Games" (on the course page as solution to Assignent 2, Problem 4). This is, in a sense, an example solution of this previous homework written up in a beautiful way.

Exercise 00 Read through my book chapter on randomized tools (available on the course page or at `http://www.mpi-inf.mpg.de/~doerr/7438_chap01.pdf`).

Omit Section 1.6 and 1.7. Almost all of this you should know already. Take note of the larger set of Chernoff bounds in Sections 1.4.3 and 1.4.4. It suffices if you know by heart the ones we covered in the lecture and roughly know that there are these other ones. Read carefully Sections 1.4.5 and 1.4.6, including the few bits we did not cover in the lecture. Notice that Lemmas 1.18 to 1.20 are a formalization of "extraction independent randomness" as discussed in the lecture. In Section 1.5, also note some stronger statements than what we proved; enjoy them, keep in mind that there is more than what we did in the lecture, but no need to learn this by heart.

Exercise 1 In the last lecture, we saw four ways of dealing with dependencies. Let us use each of them to analyse the middle phase of the rumor spreading process in complete graphs (on n vertices). More precisely: Let i_t denote the number of vertices informed after round t. Prove that if $16\log n \le i_t \le n/6$, then with probability at least $1-n^{-4}$, we have $i_{t+1} \ge 1.25 i_t$.

a) [tricky, but short] Avoiding randomness: Since the process is given, of course, you can't change it. However, you may suitably re-interpret the process such that the number of newly informed nodes can be written as sum of independent random variables.

b) [easy] Extracting independent randomness: Instead of doing things by hand, as we did in the lecture, apply now Lemma 1.18 to 1.20 from the book chapter.

c) [moderate] Negative correlation: Show that the random variables indication whether a yet uninformed vertex becomes informed in one round, are negatively correlated. This allows you to use a Chernoff bound to show strong concentration around the expectation. Note that you do not have to compute this expectation using the same variables—you may as well count/estimate how many of the informed nodes are successful in informing someone "new".

d) [easy] Azuma inequality.

General comment: The constants in the statement are quite arbitrary. If you "only" can prove a statement with weaker constants, simply do so and be happy. Also, some arguments might be used in several parts of this exercise, feel free to do so (unless this avoids you using the intended main argument; e.g., your proofs for (a) to (c) should nowhere rely on the Azuma inequality.

Exercise 2 (*tricky, but short*) Often, it is easy to compute the expectation of some quantity of interest and then one employs Chernoff/Azuma bounds to get a more precise understanding. Here, it is different...

The chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed to color the vertices in a way that no two adjacent vertices have the same color.

Let G be a random graph on n vertices (that is, you fix the n vertices and then flip a fair coin for each possible edge $\{u,v\}$ to decide whether it is present in G or not. Prove that there is an integer interval of length $O(\sqrt{n \log n})$ such that with probability $1-1/n$, this $\chi(G)$ lies in this interval.

Comment: With more advanced methods, one can even show that there are two values such that with probability $1-o(1)$, $\chi(G)$ takes one of these two values. Again without knowing the expectation!