Eric Berberich, Michael Kerber **WS 2013**

Excercises
Computational Geometry

http://www.mpi-inf.mpg.de/departments/d1/teaching/ws13/ComputationalGeometry/

Sheet 10 Deadline: 07.01.2014, 10:00am

This exercise sheet is optional; you can get up to 30 bonus points

Exercise 1 (*10 pts*)[Lower bounds for coresets]
Show that, given $\epsilon > 0$, there exists a set of points P such that any ϵ-coreset of P has size at least

$$\lceil 1 + \frac{2}{2\epsilon + \epsilon^2} \rceil \approx \frac{1}{\epsilon}$$

(this shows that the coreset construction with $\frac{2}{\epsilon}$ points is almost tight). Proceed in the following way:

a) Consider the *standard simplex* in $\mathbb{R}^{d+1}$, that is, the point set P spanned by the $d+1$ unit vectors $e_1, \ldots, e_{d+1}$. Argue that the center of the meb of P is given by $(1/(d+1), \ldots, 1/(d+1))$.

b) Next, consider the set $Q := \{e_2, \ldots, e_{d+1}\} = P \setminus \{e_1\}$. What is the meb center c_Q of Q? Compute the radius r_Q of the meb of Q and the distance δ_Q of the meb center to e_1.

c) Set $d := \lfloor 1 + \frac{2}{2\epsilon+\epsilon^2} \rceil$, show that

$$\frac{\delta_Q}{r_Q} \geq (1 + \epsilon).$$

and argue why this implies the claimed lower bound.

Exercise 2 (*10 pts*)[Improved algorithm for k-center]
Recall from the lecture that we can compute an approximate k-center clustering in time $2^{O(k \log k)/\epsilon}$ and an approximate 1-center in $O(\frac{nd}{\epsilon^2} + 1/\epsilon^5)$. Moreover, we have the following generalization of Johnson-Lindenstrauss (the proof is a relatively straight-forward combination of the original JL lemma and coresets, but it is not required here)

Theorem: For $0 < \epsilon < 1$, a set $P \subset \mathbb{R}^d$ of n points, and $m \geq C \log(n)/\epsilon^3$ for a suitable constant $C > 36$, there is a map $f : \mathbb{R}^d \to \mathbb{R}^m$ such that for any subset $S \subset P$,

$$(1-\epsilon)\mathrm{rad}(S) \leq \mathrm{rad}(f(S)) \leq (1+\epsilon)\mathrm{rad}(S),$$

where $\mathrm{rad}(\cdot)$ stands for the radius of the minimum enclosing ball. Moreover, a map $f(p) = \sqrt{d/m} \cdot \pi(p)$ where π is the projection to a m-dimensional subspace of $\mathbb{R}^d$ chosen uniformly at random, has this property with probability of at least $1/2$.

Use these facts to design an algorithm for approximate k-center which returns a correct result with a probability of at least 99% and runs in

$$n \log n 2^{O(k \log k)/\epsilon} + O(\frac{dn \log n}{\epsilon^3}).$$

(Hint: It helps to assume first that you have an oracle available that tells you a map f as in the theorem above, and to get rid of that oracle in a second step)

Exercise 3 (*10 pts*)[Kinetic alpha complexes]
We design a kinetic data structure for alpha complexes for a fixed value $\alpha > 0$ and a point set P in the plane. We use the following definitions: An edge ab is called *short* if $\|a-b\| \leq 2\alpha$. An edge ab is called *Gabriel* if the disk with $\bar{ab}$ as diameter has no point of P in its interior. Our definition from week 7 for alpha complexes is equivalent to the following property (you can use that without proof): A triangle abc belong to the alpha complex if and only if it belongs to the Delaunay triangulation and its circumradius is at most α. An edge ab belongs to the alpha complex if and only if it is short and Gabriel, or it is on the boundary of a triangle that belongs to the alpha complex.

a) Give an example of an alpha complex that contains an edge that is not Gabriel. Can you also given an example that contains a non-short edge?

b) Design certificates to monitor whether a triangle has circumradius at most α, and whether an edge has length at most 2α. What are the degrees of these certificates if the points move on linear trajectories?

c) We assume generic position, that is, no two events occur at the same time. Show that, when a *short* edge switches its Gabriel status at time t, it belongs to the alpha complex in the time interval $[t-\epsilon, t+\epsilon]$ for some sufficently small interval.

d) Describe a kinetic data structure for alpha complexes which maintains the Delaunay triangulation for P and holds flag for each edge and triangle denoting whether the object is currently in the alpha complex. Describe the type of events and how to update the kinetic data structure accordingly. Discuss the solution with respect to the quality measures introduced in the lecture.