1 Some Definitions and Examples

Uniform Matroid $(S, \mathcal{I})$ with S set of n elements, $\mathcal{I}$ family of all subsets of elements with cardinality at most k, for some given $k \in \mathbb{N}$.

Graphic Matroid $(S, \mathcal{I})$ with S set of edges of a undirected connected graph G, $\mathcal{I}$ family of all cycle-free subsets of edges (= forests).

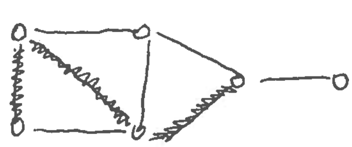
Basis:

- Independent set $I \in \mathcal{I}$ of maximum cardinality.
- Uniform matroid: Subset of size exactly k.
- Graphic matroid: Max-cardinality forest, spanning tree in G.

Bases of the graphic matroid

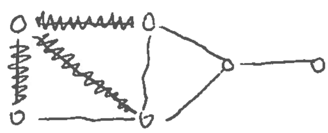
Dual matroid:

- $\mathcal{I}^*$ contains all sets $J \subseteq S$ such that there is basis $B \in \mathcal{I}$ with $B \subseteq S - J$.
- Uniform matroid: All sets J of cardinality at most $n - k$, i.e., at least k elements remain in $S - J$ (dual matroid is also a uniform matroid).
- Graphic matroid: All edge sets E' s.t. removal of E' keeps G connected, i.e., there is spanning tree of G in $E - E'$



Basis in the dual matroid

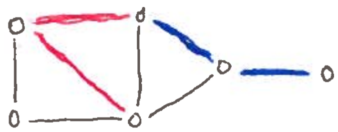
Independent set in the dual matroid, but not a basis



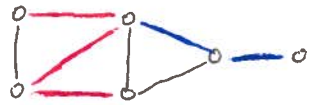
Dependent set in the dual matroid

Acceptable coloring (B, R):

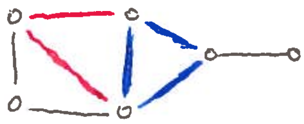
- $B \in \mathcal{I}$, $R \in \mathcal{I}^*$, $B \cap R = \emptyset$.
- Uniform matroid: B has at most k elements, R at most $n-k$ elements
- Graphic matroid: B is forest. R is such that removal of it keeps G connected.



acceptable coloring

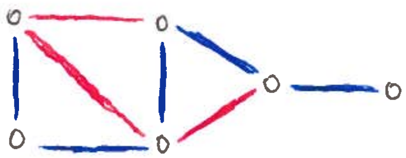


not acceptable colorings

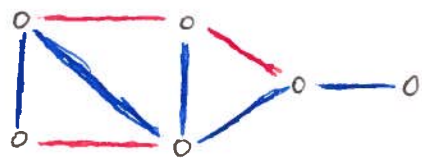


Total extension of acceptable coloring (B, R):

- Acceptable coloring (B', R') such that $B \subseteq B'$, $R \subseteq R'$ and $B \cup R = S$.
- Uniform matroid: In (B', R') there are exactly k blue elements and $n-k$ red elements, all blue elements in B are still blue, all red elements in R are still red.
- Graphic matroid: In (B', R') every edge is colored, B' is spanning tree, R' all edges except B'. All blue edges in B are still blue, all red edges in R are still red.



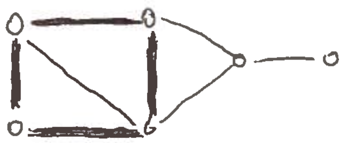
Total extension of acceptable coloring above



Total acceptable coloring, but not an extension of the above

Cycle and cut in a matroid:

- Cycle: Inclusion-minimal dependent set; Cut: Inclusion-minimal set that intersects all bases.
- Uniform matroid: Cycles are subsets of exactly $k+1$ elements, cuts are subsets of exactly $n-k+1$ elements.
- Graphic matroid: Cycles are simple cycles in G, cuts are inclusion-minimal cuts in G that every spanning tree must cross.



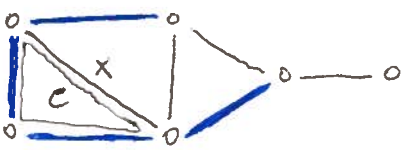
Cycle = Cycle

Cuts in the graph and in the matroid

Cuts in the graph but not in the matroid

Fundamental cycle:

- Let $B \in \mathcal{I}$, $B \cup \{x\} \notin \mathcal{I}$, so $B \cup \{x\}$ is dependent. The fundamental cycle $C \subseteq B \cup \{x\}$ is the inclusion-minimal dependent subset.
- Uniform matroid: $B \cup \{x\}$ must have cardinality exactly $k+1$, so $C = B \cup \{x\}$, since every smaller set is in $\mathcal{I}$.
- Graphic matroid: Let B be forest and $B \cup \{x\}$ not. $B \cup \{x\}$ is dependent set, it contains a cycle of the graph. There is a unique inclusion-minimal set $C \subseteq B \cup \{x\}$ that is dependent, it is the unique cycle in G closed when x is added to B.



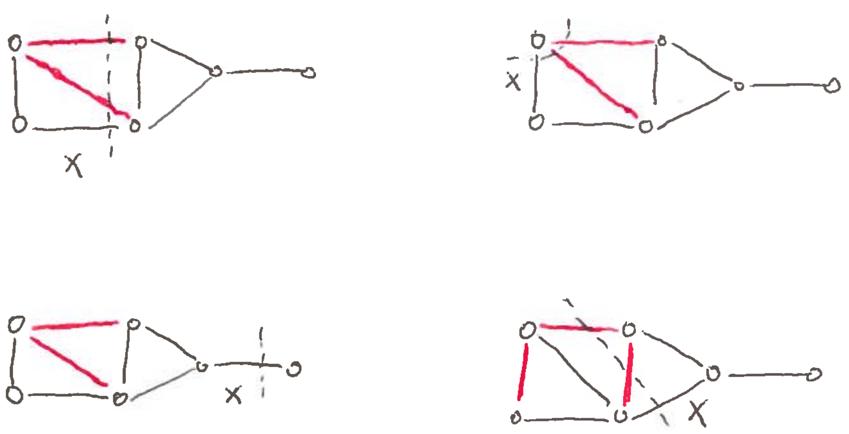
Fundamental cycle C in $B \cup \{x\}$, note B composed of blue edges is in $\mathcal{I}$

"not" = "

Also $B \cup \{x\} \supset C$, since not all blue edges of B are part of the cycle in $B \cup \{x\}$.

Fundamental cut:

- Let $R \in \mathcal{I}^*$, $R \cup \{x\} \notin \mathcal{I}^*$. Then $R \cup \{x\}$ is dependent in the dual matroid. The fundamental cut $C \subseteq R \cup \{x\}$ is the inclusion-minimal dependent set of the dual matroid.
- Uniform matroid: $|R \cup \{x\}| \geq n - k + 1$, since otherwise upon removal we would leave a basis of k elements. The fundamental cut $C = R \cup \{x\}$ since every smaller set is in $\mathcal{I}^*$.
- Graphic matroid: Let R be a set of edges that upon removal keeps G connected, and suppose that $R \cup \{x\}$ cuts the graph in several pieces. Then there is a unique inclusion-minimal set $C \subseteq R \cup \{x\}$ that cuts the graph in several pieces.



Some fundamental cuts for sets R of red edges and edges x with $R \in \mathcal{I}^*$ and $R \cup \{x\} \notin \mathcal{I}^*$