Exercise 1: It's a Colorful Life

Task 1: How the Colors Get into the Trees

1. Change the Cole-Vishkin algorithm from the lecture so that it requires only $1/2 \cdot \log^* n + \mathcal{O}(1)$ rounds. The result should still be a message passing algorithm, so don't use pointer jumping! (Hint: Compare what information can be gathered locally in T rounds to what the Cole-Vishkin algorithm actually relies on.)

2. Find a message passing algorithm that 3-colors a rooted tree, i.e., a tree in which each non-root node initially knows which neighbor is its parent, in $\log^* n + \mathcal{O}(1)$ rounds. (Hint: Leverage the same observation as for part a.)

3. Show how to $(\Delta+1)$-color a graph of maximum degree $\Delta \in \mathcal{O}(1)$ in $\log^* n + \mathcal{O}(1)$ rounds. (Hint: Decompose the graph into Δ collections of rooted trees and oriented cycles.)

Task 2: Sorry, but they just all Look alike to me!

Given a graph $G = (V, E)$, an *independent set* $I \subseteq V$ satisfies that there is no edge $e \in E$ so that $e \subseteq I$, i.e., I contains no pair of neighbors in the graph. A *maximal independent set (MIS)* is an independent set $I \subseteq V$ such that $I \cup \{v\}$ is not independent for any $v \in V \setminus I$.

Suppose that in the doubly linked list, we want to join nodes to sublists of 2 or 3 nodes under control of the same processor.

1. Show that an MIS algorithm (i.e., one that computes an MIS) can be used to construct such sublists using $O(1)$ additional rounds!

2. Show that an algorithm computing such sublists can be used to compute an MIS in $O(1)$ additional round!

3. Show that an algorithm computing a 3-coloring can be used to compute an MIS in $O(1)$ additional rounds!

4. Show that an algorithm computing an MIS can be used to compute a 3-coloring of the list in $O(1)$ additional rounds!

5. What can you infer about the time complexity of optimal algorithms for these tasks?

6.* Show that an algorithm computing an MIS on arbitrary graphs can be used to compute a $(\Delta+1)$-coloring of a graph of maximum degree Δ! (Hint: Replace each node by a clique (a.k.a. complete graph) of $\Delta+1$ nodes. Interpret each clique node as one of the possible colors of the original node. Add edges so that no adjacent cliques (original nodes) will have the same color if you compute an MIS of the new graph. Let each node simulate its entire clique in the MIS algorithm.)

Task 3*: Confusing the NSA

1. Add your name and email address to Cosmina's list for this task. You will receive your unique identifier by mail from Cosmina.

2. Be a node in the Cole-Vishkin algorithm, executed by email! All messages to your neighbors will be send to Cosmina, who will forward them. The catch: there is *no* guarantee as to when she will forward your mail or in which order! Send a termination message with your final color to Cosmina when you think you're done. She'll ignore you from then on, but that's the only way you can output something!

    - You may add some additional information to your messages to deal with this issue, but keep it as simple as possible!
    - Don't talk strategies beforehand. Where's the fun in that?
    - Agree in the exercise session on whether you'd like to reduce the number of colors to 3 for additional challenge or stop at 5!

3. Discuss the (ideally hilarious) results in the exercise session!