

Exercise 7: Lost in complexity

Task 1: Why is everything so hard?!?

In this exercise, we always consider connected, simple, weighted graphs $G = (V, E, W)$, restrict message size to $\mathcal{O}(\log n)$ bits, and assess worst-case round complexity as a function of the (hop) diameter D and n .

- Show that finding any approximation to the (weighted) distance between a given pair of nodes $s, t \in V$ takes $\Omega(\sqrt{n}/\log^2 n + D)$ rounds. (Hint: Use the same technique and graph as in the lecture, just change the weights.)
- Show that finding a Steiner tree requires $\Omega(\sqrt{n}/\log^2 n + D)$ rounds, regardless of the size of the subset T of nodes that needs to be connected to each other (unless it is 1). (Hint: Attach some irrelevant nodes to the construction from a) for $|T| \leq n/2$ and to the MST construction for $|T| > n/2$.)
- For $s, t \in V$, an $s-t$ cut is a subset $s \in S \subseteq V \setminus \{t\}$. The weight of the cut is the sum of weights of all edges $\{v, w\} \in E \cap (S \times (V \setminus S))$ crossing the cut. Show that finding any approximation to the weight of a minimum $s-t$ cut takes $\Omega(\sqrt{n}/\log^2 n + D)$ rounds.
- Conclude that finding an approximate maximum flow or even approximating the value of such a flow requires $\Omega(\sqrt{n}/\log^2 n + D)$ rounds.

Task 2: Harder Better Slower

In this exercise, we show that determining the diameter of a graph more accurately than factor $3/2$ requires $\Omega(n/\log n)$ rounds (or large messages).

- For a set disjointness instance (x, y) , construct a graph with $\mathcal{O}(\sqrt{N})$ nodes that has diameter 2 if x and y encode disjoint sets, so that there is a cut with $\mathcal{O}(\sqrt{N})$ edges between the parts encoding x and y , respectively. (Hint: Start from k pairs of nodes (l_j, r_j) , each connected by an edge. Now add edges so that there is a path of length 2 from l_j to r_k if $x_{jk} = 0$ or $y_{jk} = 0$, but not if $x_{jk} = y_{jk} = 1$. Finally, add two nodes and some edges to make sure that the diameter is 2 if x and y encode disjoint sets and 3 otherwise.)
- Show that Alice and Bob can simulate a distributed algorithm that uses B -bit messages to compute (or approximate) the diameter of such a graph in T rounds, with a total communication complexity of $\mathcal{O}(\sqrt{N}BT)$.
- Conclude that $T \in \Omega(\sqrt{N}/B)$ in the worst case, no matter what algorithm is used. Specifically, follow that if $B \in \mathcal{O}(\log n)$, it requires $\Omega(n/\log n)$ rounds to determine the diameter of a graph more accurately than up to factor $3/2$.

Task 3*: Be more constructive!

- Check up on the prime number theorem!
- Show that for any $k \in \mathbb{N}$ and any constant $C \in \mathbb{N}$, the number of primes in the range $[2^k, 2^{k+C}]$ is in $2^{\Theta(k+C)}/k$.

- c) Prove that for an N -bit number, the number of different $\Theta(\log N)$ -bit primes that divides it is bounded by $\Theta(N/\log N)$. Use this to find suitable choices of k and C such that the number of primes in the range $[2^k, 2^{k+C}]$ is polynomial in N and the probability that, for a fixed N -bit number, a uniformly random prime from this range divides it is at most $N^{-\Theta(1)}$.
- d) Check up on the AKS primality test!
- e) Infer that there is a protocol solving equality with error probability $N^{-\Theta(1)}$ that uses private randomness, communicates $\mathcal{O}(\log N)$ bits, and requires only polynomial computations, both for construction and execution!
- f) Check up on your ability to explain this to others in the exercise session!