(1) Gram-Schmidt Orthogonalization [4 points]

a. Calculate an orthogonal basis for $\mathbb{R}^3$ from the vectors

\[
v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}\]

using Gram-Schmidt Orthogonalization.

b. Calculate an orthogonal basis for the functional basis $[1, x, x^2]$ on the interval $[0, 1]$ using Gram-Schmidt Orthogonalization. (Remember to use the inner product for function spaces)

c. Sketch the resulting functions from b.

(2) Eigenvectors and –values in $\mathbb{R}^3$ [8 points]

a. Show that $u$ is an eigenvector of the matrix $uu^T$ and has an eigenvalue of $||u||^2$.

b. Show that $uu^T$ has only one non-zero eigenvalue.

c. Given a vector $v$ with $||v||^2 = 1$ show that $I - vv^T$ has two non-zero eigenvalues with value 1. (Hint: What are eigenvectors of $I$?)

d. Given eigenvectors $e_1, e_2, e_3 \in \mathbb{R}^3$ with corresponding eigenvalues $d_1, d_2, d_3$, how do you reconstruct the source matrix $M$ from which they were calculated?

(3) Integral transformation [4 points]

Show that the surface area $A$ of a sphere given by $f(x, y) = \pm \sqrt{r^2 - x^2 - y^2}$ is given by the formula $A = 4\pi r^2$. (Hint: Transform the integral to polar coordinates)

(4) Curvature [4 points]

a. Derive the curvature function $\kappa(t)$ for the following functions:

\[
f_1(t) = \begin{pmatrix} r \cos(t) \\ r \sin(t) \end{pmatrix}, \quad f_2(t) = \begin{pmatrix} \frac{1}{t} \end{pmatrix}\]

b. Given the surface $f(x, y) = 3 + xy$, what is the curvature $\kappa(\alpha)$ at point $\left(\frac{3}{3}, 0\right)$ in direction $\alpha$ (use polar coordinates)? In which direction is it minimal / maximal?