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Geometric Modeling

Assignment sheet 11 (Subdivision, Implicit Surfaces, and Topology, due July 10th 2008)

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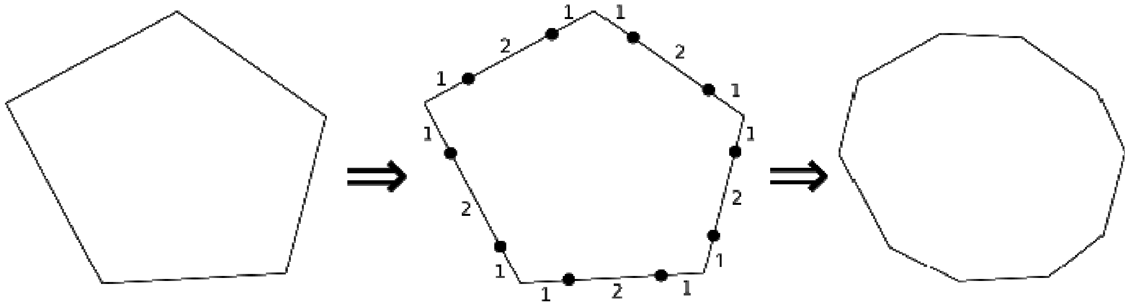
(1) Chaikin's Corner Cutting [3 points]
Consider a closed control polygon. Chaikin's algorithm can be formulated as subdividing each linear segment 1:2:1 and using the arising points as the control points of the refined control polygon (cf. illustration). Show that the limit curve is a C1-continuous, piecewise quadratic Bézier curve.
Hint: Remember the De Casteljau algorithm.



(2) Wavelet Compression [2 points]
Consider an orthonormal Wavelet basis $u_1(x),\ldots,u_m(x)$ ($\langle u_i \mid u_j \rangle = \delta_{ij}$) and let $c_1,\ldots,c_m$ be coefficients such that $f(x)=\sum_{i=1}^{m} c_i u_i(x)$. Let $\pi(i)$ be a permutation of $1,\ldots,m$ and $\hat{f}(x)=\sum_{i=1}^{\hat{m}} c_{\pi(i)} u_{\pi(i)}(x)$ be the approximation to f produced by omitting the last $m-\hat{m}$ coefficients with respect to π. Show that for a given $\hat{m}$, π minimizes the squared error $\left\| f(x)-\hat{f}(x) \right\|^2 = \left\langle f(x)-\hat{f}(x) \mid f(x)-\hat{f}(x) \right\rangle$ if it sorts the c_i by decreasing magnitude.

(3) Marching Squares [4 points]
Consider the function $f(x,y)=4x^2y-y^2-2x^2+0.25$

a. Sketch the isocontour $f(x,y)=0$ over $(x,y)\in(0,1)\times(0,1)$. Mark positive and negative regions.

b. Evaluate f at (0,0), (0,1), (1,0) and (1,1). Sketch all possible lines that a first order accurate marching squares algorithm would produce. Does any one of them correspond to the topology of the true isocontour, as found in (a)?

c. *Structurally unstable* cases are destroyed by an arbitrarily small perturbation and often neglected by standard algorithms. Show that in bilinearly interpolated fields, self-intersecting isolines are structurally unstable.
Hint: What properties do the four scalars s_1, s_2, s_3, s_4 at the corners (0,0), (0,1) (1,0) (1,1) of the unit square need to have such that a self-intersection within $(0,1)\times(0,1)$ can occur? What type of function results from their bilinear interpolation? When do isolines cross, and what happens if you slightly perturb the isovalue or any of the s_i?

(4) Metric Spaces and Open Sets [3 points]

a. Prove the following theorem: If two metrics d_1 and d_2 on the same set X have the property that for any $\varepsilon>0$, there exists a $\delta>0$ such that $d_1(x,y)<\delta \Rightarrow d_2(x,y)<\varepsilon$ and $d_2(x,y)<\delta \Rightarrow d_1(x,y)<\varepsilon$, then these metrics define the same open sets in X.

b. Use the theorem from (a) to show that a function $f:\mathbb{R}^n\rightarrow\mathbb{R}^k$ which is ε-δ-continuous with respect to any single one of the following metrics is continuous with respect to all of them:

$$d_1(x,y)=\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} \qquad d_2(x,y)=\sum_{i=1}^{n}|x_i-y_i| \qquad d_3(x,y)=\max|x_i-y_i|$$

(5) Complexes [5 points]

a. In a triangulated surface, let V_n denote the number of vertices which have degree *n* (i.e., at which *n* edges meet). Consider a triangulation of the sphere in which all vertices have either degree 5 or 6. What are the possible values of V_5?

b. Consider a regular triangulation of a torus, i.e., the same number *n* of triangles meet at each vertex. What are the possible values of *n*?

c. Consider a closed surface in which each face is a pentagon and four faces meet at each vertex. Show that if the number of faces is not a multiple of 8, then the surface is not orientable.

(6) Stars and Links [3 points]
Let K be a simplicial 2-complex that triangulates the closed disk. Let *a* and *b* be interior vertices, *u* and *v* be boundary vertices. Let *ab* be an interior edge, *uv* a boundary edge. Draw K such that it contains (among others) the specified vertices and edges. Then, draw the star and link of the following subsets: {*a*}, {*ab*}, {*a*, *b*, *ab*}, {*u*, *v*, *uv*}.
You may use colors to distinguish the star from the link, but please make a separate sketch for each of the four subsets. Make sure to clearly mark every vertex, edge and face that belongs to a star or a link!