(1) Chaikin’s Corner Cutting [3 points]
Consider a closed control polygon. Chaikin’s algorithm can be formulated as subdividing each linear segment 1:2:1 and using the arising points as the control points of the refined control polygon (cf. illustration). Show that the limit curve is a C1-continuous, piecewise quadratic Bézier curve.

*Hint:* Remember the De Casteljau algorithm.

(2) Wavelet Compression [2 points]
Consider an orthonormal Wavelet basis \( u_1(x), \ldots, u_m(x) \) (\( \langle u_i \mid u_j \rangle = \delta_{ij} \)) and let \( c_1, \ldots, c_m \) be coefficients such that \( f(x) = \sum_{i=1}^{m} c_i u_i(x) \). Let \( \pi(i) \) be a permutation of \( 1, \ldots, m \) and \( \hat{f}(x) = \sum_{i=1}^{\hat{m}} c_{\pi(i)} u_{\pi(i)}(x) \) be the approximation to \( f \) produced by omitting the last \( m - \hat{m} \) coefficients with respect to \( \pi \). Show that for a given \( \hat{m} \), \( \pi \) minimizes the squared error \( \| f(x) - \hat{f}(x) \| = \langle f(x) - \hat{f}(x) \mid f(x) - \hat{f}(x) \rangle \) if it sorts the \( c_i \) by decreasing magnitude.

(3) Marching Squares [4 points]
Consider the function \( f(x,y) = 4x^2 - y^2 - 2x^2 + 0.25 \)

a. Sketch the isocontour \( f(x,y) = 0 \) over \((x,y) \in (0,1) \times (0,1)\). Mark positive and negative regions.

b. Evaluate \( f \) at \((0,0), (0,1), (1,0) \) and \((1,1)\). Sketch all possible lines that a first order accurate marching squares algorithm would produce. Does any one of them correspond to the topology of the true isocontour, as found in (a)?
c. **Structurally unstable** cases are destroyed by an arbitrarily small perturbation and often neglected by standard algorithms. Show that in bilinearly interpolated fields, self-intersecting isolines are structurally unstable.

**Hint:** What properties do the four scalars $s_1$, $s_2$, $s_3$, $s_4$ at the corners $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ of the unit square need to have such that a self-intersection within $(0,1) \times (0,1)$ can occur? What type of function results from their bilinear interpolation? When do isolines cross, and what happens if you slightly perturb the isovalue or any of the $s_i$?

(4) Metric Spaces and Open Sets [3 points]

a. Prove the following theorem: If two metrics $d_1$ and $d_2$ on the same set $X$ have the property that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_1(x,y) < \delta \Rightarrow d_2(x,y) < \varepsilon$ and $d_2(x,y) < \delta \Rightarrow d_1(x,y) < \varepsilon$, then these metrics define the same open sets in $X$.

b. Use the theorem from (a) to show that a function $f: \mathbb{R}^n \rightarrow \mathbb{R}^k$ which is $\varepsilon$-$\delta$-continuous with respect to any single one of the following metrics is continuous with respect to all of them:

$$d_1(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

$$d_2(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

$$d_3(x,y) = \max |x_i - y_i|$$

(5) Complexes [5 points]

a. In a triangulated surface, let $V_n$ denote the number of vertices which have degree $n$ (i.e., at which $n$ edges meet). Consider a triangulation of the sphere in which all vertices have either degree 5 or 6. What are the possible values of $V_5$?

b. Consider a regular triangulation of a torus, i.e., the same number $n$ of triangles meet at each vertex. What are the possible values of $n$?

c. Consider a closed surface in which each face is a pentagon and four faces meet at each vertex. Show that if the number of faces is not a multiple of 8, then the surface is not orientable.

(6) Stars and Links [3 points]

Let $K$ be a simplicial 2-complex that triangulates the closed disk. Let $a$ and $b$ be interior vertices, $u$ and $v$ be boundary vertices. Let $ab$ be an interior edge, $uv$ a boundary edge. Draw $K$ such that it contains (among others) the specified vertices and edges. Then, draw the star and link of the following subsets: $\{a\}$, $\{ab\}$, $\{a, b, ab\}$, $\{u, v, uv\}$.

You may use colors to distinguish the star from the link, but please make a separate sketch for each of the four subsets. Make sure to clearly mark every vertex, edge and face that belongs to a star or a link!