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Geometric Modeling

Assignment sheet 3 (Interpolation/Approximation, due May 13th 2008)

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(1) Interpolation [7 points]

Let $\{(x_i, y_i) | i = 1 \dots n\}$ be a set of given points. To interpolate them a polynomial of degree n-1 is necessary in general. Derive a formula $L(x)$ for such a function which passes exactly through all the given points.

a. Develop polynomial basis functions $L_k(x)$ of degree n-1 with the following property:

$$L_k(x_j) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \quad k = 1 \dots n$$

Hints: How does the function $\frac{(x-a)}{(b-a)}$ behave?
Use a product of such functions.
Which x_i should be omitted to achieve the correct degree?

b. Use the basis functions to express the interpolating polynomial $L(x)$. (Hint: sum up the correctly weighted y-entries).

c. Test your formula using the data points sampled from the one dimensional function $f(x) = \sqrt{x}$:

$$f(1) = 1 \quad f(4) = 2 \quad f(16) = 4$$

Use a plotting tool to display this function and the "original curve" $f(x)$ in the same coordinate system and provide a printout in your solution. Also sketch the error function $e(x) = |f(x) - L(x)|$

(2) Compactly Supported Basis Functions [8 points]

a. We want to interpolate n regularly spaced points (x_i, y_i) , $i = 1 \dots n-1$ with $x_i = ih$ sampled from a 1D function $f(x) = y$. To avoid oscillation artifacts, we want to use *piecewise* polynomial basis functions $b_i(x)$ which have the following properties :

$$b_i(x) = \begin{cases} 1 & x = ih \\ 0 & x \leq (i-1)h \\ 0 & x \geq (i+1)h \end{cases} \qquad b_i \in \mathbb{C}^1, i = 1 \dots n-2$$

Derive basis functions b_i which satisfy the properties.

b. Use a plotting tool to draw the function you get by interpolating the sample points

$$s(0) = 1 \quad , \quad s(1) = 2 \quad , \quad s(2) = 3 \quad , \quad s(3) = 4$$

on the interval $[0,3]$ using the basis functions derived in (a) with $h = 1$.

c. How do you need to change the basis functions b_i to make them reproduce constant functions $f(x) = c$ faithfully not only on the sample points but on the whole interval? Write down the new basis functions.

(3) Least-Squares Approximation [4 points]

Given n sample points $\binom{x_0}{y_0} \dots \binom{x_n}{y_n} \in \mathbb{R}^2$, how do you find the center $\binom{a}{b}$ and radius r of the best fitting circle in an algebraic sense?

Hint: Represent the circle as an implicit function.
Substituting $c = a^2 + b^2 - r^2$ may be helpful to reduce the degree of the error function (this means effectively solving for (a,b,c) instead of (a,b,r).

(4) Principal Component Analysis [1 points]

Sketch the normal, tangent and the PCA ellipsoid for the marked points and their neighbors in the following 3 diagrams (no calculations necessary, a rough sketch is sufficient) :