(1) Bézier Curves [4 points]

Find a cubic Bézier curve \( P(u) \), \( P : [0,1] \rightarrow \mathbb{R}^2 \) with:

\[
P(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad P(1) = \begin{pmatrix} 9 \\ 0 \end{pmatrix}
\]

which intersects itself at \( P\left(\frac{1}{4}\right) = P\left(\frac{3}{4}\right) \) orthogonally.

(2) De Casteljau algorithm and subdivision [1+2+2 points]

Given the cubic polynomial curve

\[
P(u) = \left( \frac{7}{8} \right) u^3 + \left( \frac{9}{15/4} \right) u^2 - \left( \frac{57/2}{9/2} \right) u + \left( \frac{30}{-1} \right)
\]

(a) Find the polar form \( p(u_1, u_2, u_3) \) of \( P(u) \), as well as the Bézier points (the vertices of the control polygon) \( P_0, P_1, P_2, P_3 \) of \( P(u) \) w.r.t. the interval \([2,4]\). Sketch the the control polygon. (Hint: use a full A4 paper and a meaningful scale)

(b) Evaluate the polynomial \( P(u) \) using the De Casteljau algorithm at the sample points \( u \in \{5/2, 3, 7/2\} \) and draw it into the same graph.

(c) Use the result from (b) for subdividing \( P(u) \) at \( u = 3 \) and subdivide the right part of the curve again at its midpoint \( u = 7/2 \). Add this control polygon to the same graph like before and sketch the curve described by \( P(u) \).
(3) Polar forms and derivatives [1+2+4 points]

Given is the cubic polynomial curve

\[ F(u) = \left( \frac{15}{-6} \right) u^3 + \left( \frac{27}{10} \right) u^2 - \left( \frac{9}{9} \right) u \]

w.r.t. the parameter interval [0,1].

(a) Find the first and second derivative of \( F \).

(b) Find the polar form \( f(u_1,u_2,u_3) \) of \( F \) as well as the polar forms of the derivatives \( F' \) and \( F'' \). Show that they are equal to

\[ 3 f(u_1,u_2, \bar{1}) \text{ and } 6f(u_1, \bar{1}, \bar{1}) \]

respectively.

Note: \( f(u_1,u_2, \bar{1}) \) is short for \( f(u_1,u_2,1)-f(u_1,u_2,0) \)

(c) Prove that the curvature of a Bézier curve at the starting point \( P_0 \) is given by:

\[ \kappa^2(P_0) = 2 \frac{n-1}{n} \frac{\text{area}(P_0,P_1,P_2)}{\text{dist}^3(P_0,P_1)} = \frac{n-1}{n} \frac{h}{a^2} \]
(4) DeBoor algorithm [2+2 points]

a. Given the uniform B-spline defined by the points
   \[ P_0 = \begin{pmatrix} -2 \\ -10 \end{pmatrix}, \quad P_1 = \begin{pmatrix} -4 \\ 2 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 4 \\ -7 \end{pmatrix} \]
   and the knot vector \([0,1,2,3,4,5]\). Evaluate the position of the curve at parameter \(t=2.5\) using DeBoor's algorithm. Sketch the control polygon and the points constructed by the algorithm.

b. For the B-spline from (a), compute the corresponding Bézier control points which describe the same cubic curve. Sketch the points and the resulting Bézier curve.