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Geometric Modeling

Assignment sheet 7 (Rational Curves/Surfaces, due June 10th 2008)

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(1) Reparametrization of Rational Bézier Curves [4 points]

a. Let $\gamma > 0$. Prove that $\phi(t) := \dfrac{\gamma t}{1-(1-\gamma)t}$ is a parameter transformation of the interval [0,1], i.e. $\phi(0) = 0$, $\phi(1) = 1$, and $\phi : [0,1] \to [0,1]$ bijective.

b. Given is a rational Bézier curve *F* with control points $b_0,b_1,\ldots,b_n$ and weights $w_0,w_1,\ldots,w_n$. In addition, let $\widetilde{F}(t) := F(\phi(t))$ with $\phi(t)$ as defined in (a). Prove that $\widetilde{F}$ is a rational Bézier curve as well and find its control points $\widetilde{b}_i$ and weights $\widetilde{w}_i$.
Hint: Consider the parameter transformation ϕ for a Bernstein polynomial B_i^n first.

c. Prove that any rational Bézier curve can be normalized by reparametrization such that $w_0=w_n=1$.

(2) Spline Representation of Circles [3 points]
Prove that a circle $\{(x, y) \in \Re^2 : x^2 + y^2 = r^2\}$ cannot be represented as a non-rational polynomial B-Spline curve. Why is your proof not applicable for rational splines?
Hint: The proof can be done by induction.

(3) De Casteljau Algorithm for Bézier Triangles [3 points]
A quadratic Bézier triangle is given by the control points

$$F(a,a)=\begin{pmatrix}0\\0\\0\end{pmatrix}\quad F(a,b)=\begin{pmatrix}2\\2\\4\end{pmatrix}\quad F(a,c)=\begin{pmatrix}4\\-2\\6\end{pmatrix}$$

$$F(b,b)=\begin{pmatrix}4\\4\\0\end{pmatrix}\quad F(b,c)=\begin{pmatrix}8\\0\\4\end{pmatrix}\quad F(c,c)=\begin{pmatrix}6\\-4\\4\end{pmatrix}$$

at parameter positions a=(0,0), b=(1,0), c=(0.5,1).
Which of the parameter pairs p_1=(0.25,0.5), p_2=(0.3,0.75), p_3=(0.5,0.5) is outside the triangle?
For the other parameter pairs p, evaluate the surface F(p,p) using the De Casteljau Algorithm for Bézier triangles.

(4) Fundamental Forms and Curvature [3 points]

a. Find the coefficients of the second fundamental form of the surface

$$r(u,v) = \begin{pmatrix} \cos v - u \sin v \\ \sin v + u \cos v \\ u + v \end{pmatrix}$$

b. Find the Gaussian curvature of the surface

$$r(u,v) = \begin{pmatrix} -u \\ 2 \sin v \\ 2 \cos v \end{pmatrix}$$

(5) Ruled and Developable Surfaces [7 points]

Definition : Ruled surface

A *ruled surface* can be described (at least locally) as the set of points swept by a moving straight line.

Let $p(t)$ be a given space curve and $a(t)$ be a unit vector of the sweeping line. Then the ruled surface $r(u,v)$ is given as :

$$r(u,v) = p(u) + va(u) \qquad \|a(u)\| = 1$$

The Gaussian curvature on a ruled surface is ≤ 0 everywhere.

a. Present a surface which is a ruled surface *and* a surface of revolution at the same time. Justify your answer.

b. Is the surface given by $z = xy + 3$ a ruled surface?

c. Given a space curve $r(s)$ parameterized by its arc-length s, consider a surface generated by all the tangents to the curve. Is it a developable surface? Justify your answer.
Hint: *Write down the surface equation as a ruled surface and determine the Gaussian curvature using fundamental forms.*