Advanced Course Computer Science

Music Processing

Summer Term 2009

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Signals and Fourier Transform

UNIVERSITÄT DES SAARLANDES

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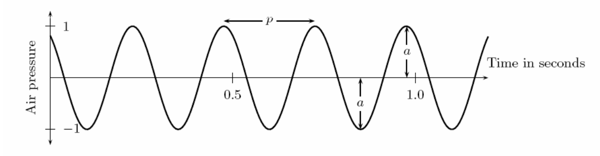
Signals

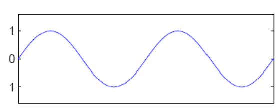
Sinusoidal



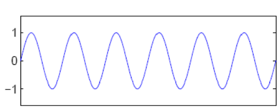
Signals

Sinusoidal $f(t) = A\sin(2\pi(\omega t - \varphi))$ for $t \in [0,2]$

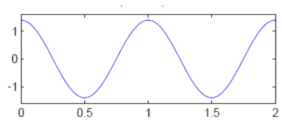
A = 1, ω = 1, φ = 0



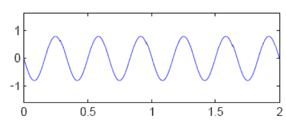
A = 1, ω = 3, φ = 0



A = 1.4, ω = 1, φ = 0.25

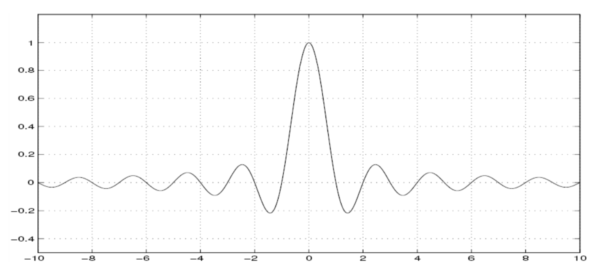


A = 0.8, ω = 3, φ = 0.5



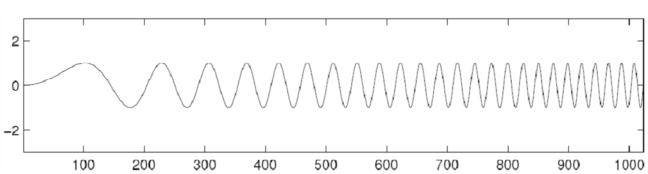
Signals

Sinc-function $\operatorname{sinc}(t) := \begin{cases} \frac{\sin \pi t}{\pi t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$



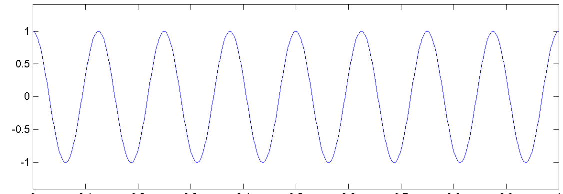
Signals

Chirp signal



Sampling

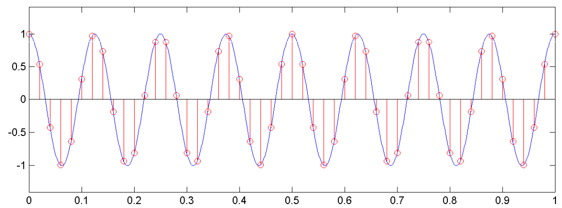
Original CT signal



Sampling

Original CT signal

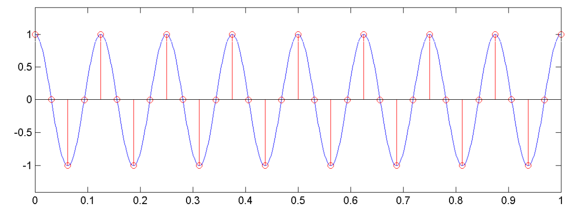
DT signal sampled with 50 Hz



Sampling

Original CT signal

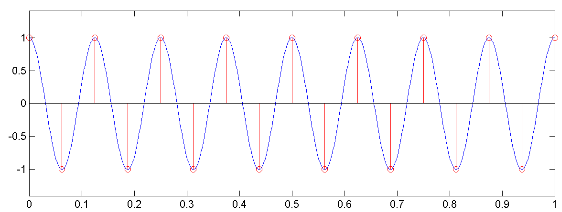
DT signal sampled with 32 Hz



Sampling

Original CT signal

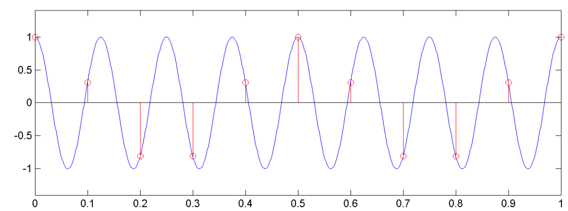
DT signal sampled with 16 Hz



Sampling

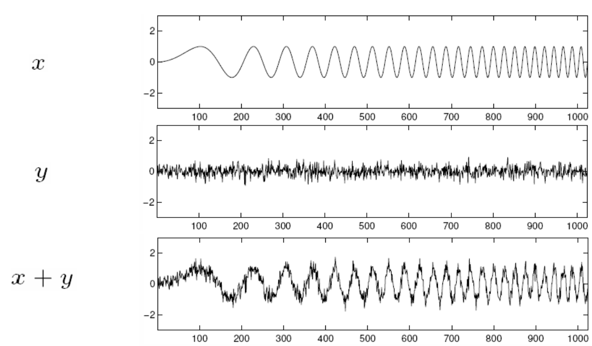
Original CT signal

DT signal sampled with 10 Hz



Signals

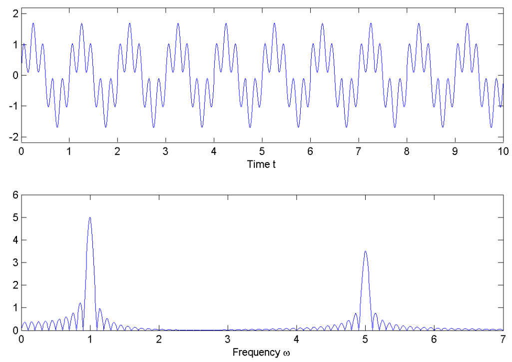
Superposition



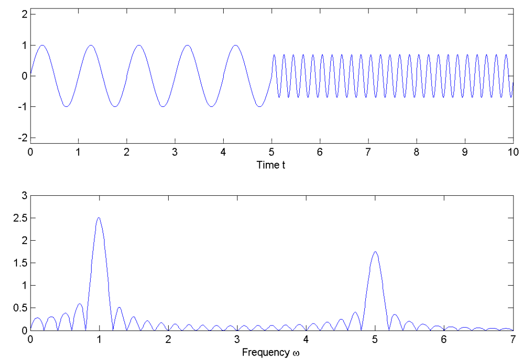
Fourier Transform

Signal space	$L^2(\mathbb{R})$	$L^2([0,1])$	$\ell^2(\mathbb{Z})$
Inner product	$\langle f\|g\rangle = \int_{t\in\mathbb{R}} f(t)\overline{g(t)}dt$	$\langle f\|g\rangle = \int_{t\in[0,1]} f(t)\overline{g(t)}dt$	$\langle f\|g\rangle = \sum_{n\in\mathbb{Z}} x(n)\overline{y(n)}$
Norm	$\\|f\\|_2 = \langle f\|f\rangle^{\frac{1}{2}}$	$\\|f\\|_2 = \langle f\|f\rangle^{\frac{1}{2}}$	$\\|x\\|_2 = \langle x\|x\rangle^{\frac{1}{2}}$
Definition	$L^2(\mathbb{R}) :=$ $\{f : \mathbb{R} \to \mathbb{C} \mid \\|f\\|_2 < \infty\}$	$L^2([0,1]) :=$ $\{f : [0,1] \to \mathbb{C} \mid \\|f\\|_2 < \infty\}$	$L^2(\mathbb{Z}) :=$ $\{f : \mathbb{Z} \to \mathbb{C} \mid \\|x\\|_2 < \infty\}$
Elementary frequency function	$\mathbb{R} \to \mathbb{C}$ $t \mapsto e^{2\pi i\omega t}$	$[0,1] \to \mathbb{C}$ $t \mapsto e^{2\pi ikt}$	$\mathbb{Z} \to \mathbb{C}$ $n \mapsto e^{2\pi i\omega n}$
Frequency parameter	$\omega \in \mathbb{R}$	$k \in \mathbb{Z}$	$\omega \in [0,1]$
Fourier representation	$f(t) = \int_{\omega\in\mathbb{R}} c_\omega e^{2\pi i\omega t}d\omega$	$f(t) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi ikt}$	$x(n) = \int_{\omega\in[0,1]} c_\omega e^{2\pi i\omega n}d\omega$
Fourier transform	$\hat{f} : \mathbb{R} \to \mathbb{C}$ $\hat{f}(\omega) = \int_{t\in\mathbb{R}} f(t)e^{-2\pi i\omega t}dt$ $c_\omega = \hat{f}(\omega)$	$\hat{f} : \mathbb{Z} \to \mathbb{C}$ $\hat{f}(k) = \int_{t\in[0,1]} f(t)e^{-2\pi ikt}dt$ $c_k = \hat{f}(k)$	$\hat{x} : [0,1] \to \mathbb{C}$ $\hat{x}(\omega) = \sum_{n\in\mathbb{Z}} x(n)e^{-2\pi i\omega n}$ $c_\omega = \hat{x}(\omega)$

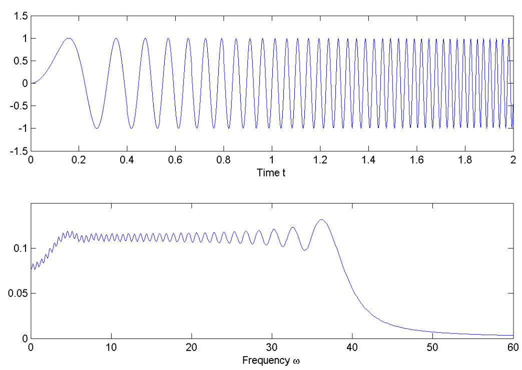
Fourier Transform



Fourier Transform

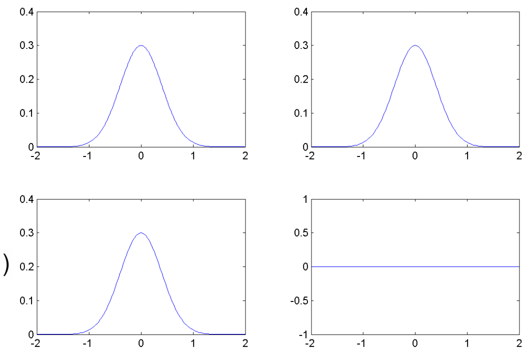


Fourier Transform



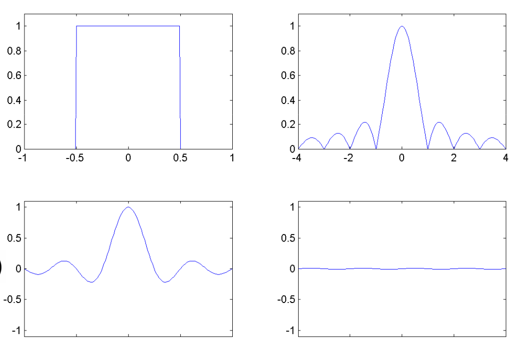
Fourier Transform

Gaussian function $f(t) = (2\pi)^{-\frac{1}{2}}\pi^{-\frac{1}{4}}e^{-\pi t^2}$



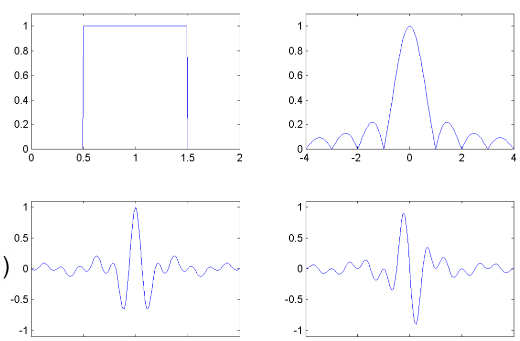
Fourier Transform

Box function



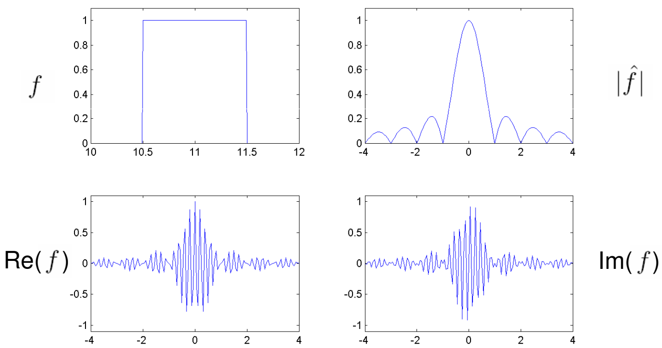
Fourier Transform

Box function (translated)



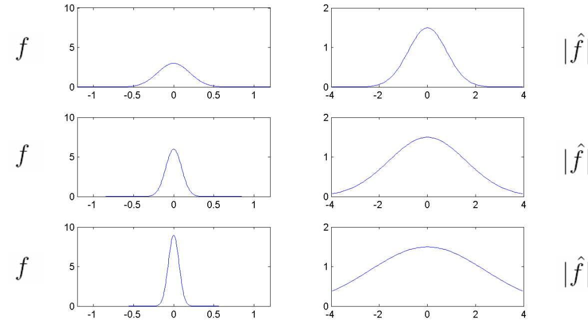
Fourier Transform

Box function (translated)



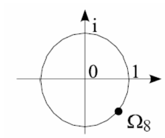
Fourier Transform

Dirac sequence



Discrete Fourier Transform (DFT)

$\Omega_N := e^{-2\pi i/N}$

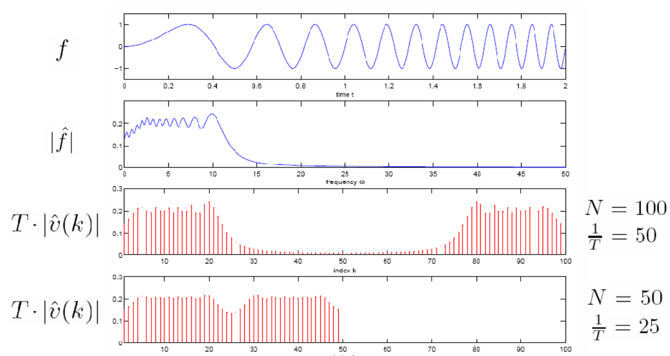


$$\mathrm{DFT}_N := \frac{1}{\sqrt{N}} \left(\Omega_N^{kj}\right)_{0<k,j<N}$$

$$= \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & \Omega_N & \cdots & \Omega_N^{(N-2)} & \Omega_N^{(N-1)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \Omega_N^{(N-2)} & \ddots & \Omega_N^{(N-2)(N-2)} & \Omega_N^{(N-2)(N-1)} \\ 1 & \Omega_N^{(N-1)} & \cdots & \Omega_N^{(N-1)(N-2)} & \Omega_N^{(N-1)(N-1)} \end{pmatrix}$$

Discrete Fourier Transform (DFT)

$v(k) = f(Tk), \quad k \in [0 : N-1]\,, \ \hat{v} = \mathrm{DFT}_N(v)$



Fast Fourier Transform (FFT)

$N = 2M$

$$\mathrm{DFT}_N \cdot \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{pmatrix} = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \mathrm{id}_M & \Delta_M \\ \mathrm{id}_M & -\Delta_M \end{array}\right) \left(\begin{array}{c|c} \mathrm{DFT}_M & 0 \\ 0 & \mathrm{DFT}_M \end{array}\right) \begin{pmatrix} v_0 \\ v_2 \\ \vdots \\ v_{N-2} \\ \hline v_1 \\ v_3 \\ \vdots \\ v_{N-1} \end{pmatrix}$$

$\mathrm{id}_M = \mathrm{diag}\,(1, 1, \ldots, 1)$

$\Delta_M = \mathrm{diag}\,(1, \Omega_N, \ldots, \Omega_N^{M-1})$