(1) Curvature [5 points]

a. Derive the curvature function \( \kappa(t) \) for the following functions:
\[
\begin{align*}
f_1(t) &= \left( r \cos t, r \sin t \right) \\
f_2(t) &= \left( t, t^2 \right) \\
f_3(t) &= \left( \cos t, \sin t \right)
\end{align*}
\]

b. Given the surface \( f(x, y) = 3 + xy \), what is the curvature \( \kappa(\alpha) \) at point \( (0, 0) \) in direction \( \alpha \) (use polar coordinates)? In which direction is it minimal / maximal?

(2) Compactly Supported Basis Functions [8 points]

a. We want to interpolate \( n \) regularly spaced points \( (x_i, y_i), i = 1 \ldots n - 1 \) with \( x_i = ih \) sampled from a 1D function \( f(x) = y \). To avoid oscillation artifacts, we want to use piecewise polynomial basis functions \( b_i(x) \) which have the following properties:
\[
b_i(x) = \begin{cases}
1 & x = ih \\
0 & x \leq (i-1)h \\
0 & x \geq (i+1)h
\end{cases}
\]
\( b_i \in \mathbb{C}, i = 1 \ldots n-2 \)

Derive basis functions \( b_i \) which satisfy the properties.

b. Use a plotting tool to draw the function you get by interpolating the sample points 
\[
s(0) = 1, \quad s(1) = 2, \quad s(2) = 3, \quad s(3) = 4
\]
on the interval \([0,3]\) using the basis functions derived in (a) with \( h = 1 \).

c. How do you need to change the basis functions \( b_i \) to make them reproduce constant functions \( f(x) = c \) faithfully not only on the sample points but on the whole interval? Write down the new basis functions.

(3) Least-Squares Approximation [4 points]

Given \( n \) sample points \( \left( x_0, y_0 \right), \ldots, \left( x_n, y_n \right) \in \mathbb{R}^2 \), how do you find the center \( \left( a, b \right) \) and radius \( r \) of the best fitting circle in an algebraic sense?

Hint: Represent the circle as an implicit function.
Substituting \( c = a^2 + b^2 - r^2 \) may be helpful to reduce the degree of the error function (this means effectively solving for \( (a,b,c) \) instead of \( (a,b,r) \).
(4) Bernstein Polynomials [2 points]

Prove the following two identities for Bernstein Polynomials:

\[ x^n = \sum_{i=k}^{n} \binom{n}{i} B_i^n(x) , \quad \frac{d}{dx} B_i^n(x) = n \left( B_{i-1}^{n-1}(x) - B_i^{n-1}(x) \right) \]

(5) Principal Component Analysis [1 point]

Sketch the normal, tangent and the PCA ellipsoid for the marked points and their neighbours in the following 3 diagrams (no calculations necessary, a rough sketch is sufficient):

![Diagram 1](image1)

![Diagram 2](image2)

![Diagram 3](image3)