Geometric Modeling 2012
Preparatory Exam

UNIVERSITÄT DES SAARLANDES

CLUSTER OF EXCELLENCE

General Instructions: Please read the following carefully before proceeding!

- You have **2h time** to finish the exam.
- Please write **your full name** and your **student registration** number on **every sheet of paper** you use to write down solutions. Write the information in the upper right corner. Do this first before writing down any solutions. Sheets without this might possibly not be taken into account for grading.
- Please **start a new page** for **each separate assignment**.
- You can answer the questions in either English or German.
- Most assignments specify how detailed the answer should be to solve it (*marked in italics*). When nothing is specified explicitly, it should be clear from the question itself.
- If no explanation is required, you might optionally still write down a few additional keywords or half a sentence so that we can see that you were on the right track even if the result might be wrong (due to a wrong calculation, for example). This might save some points that are lost otherwise, but this is not mandatory.
- You can obtain 100 points. If you get at least 50 points you will pass the exam. Bonus points from the homework are not taken into account for passing, only to improve the grade in case of passing.
- **Good luck!**

Assignment #1: Differential Geometry and Parametric Surfaces **(25 points)**

Consider the following parametric surface, which is defined for an arbitrary real constant *r*:

$$\mathbf{f} : [0,2\pi) \times [0,1] \to \mathbb{R}^3, \quad \mathbf{f}(u,v) := \begin{pmatrix} r\cos u \\ r\sin u \\ v \end{pmatrix}, \quad r > 0$$

(a) What kind of object is this (*no explanation required*)? (1 point)

(b) What are the circles of curvature at any point (u,v) in u and v-direction? Determine the center and the radius of the osculating circle at point $(u,v) = (0,0)$, if possible. (*one sentence each*) (3 points)

(c) What is the Gaussian and mean curvature at any point (u,v)? (*no explanation*) (2 points)

(d) Compute the first fundamental form. (*calculation only*) (5 points)

(e) Under which conditions is the parametrization (i.e., the mapping **f,** from the domain to the image) isometric? (*one sentence explanation*)? Provide a parametrization that is always isometric (this might involve adjusting the parametrization domain). (5 points)

(f) Is this object a developable surface? (*one sentence*) (1 points)

(g) Construct this object out of rational, quadratic Bezier patches. Specify patches listing their control points with weights. The solution might be symmetric; in this case it is sufficient to specify one piece and how to construct further pieces from symmetry. (*no further explanation necessary; if you do not remember the exact weights, you can still get up to 6 points by specifying constants and their relations symbolically.*) (8 points)

Assignment #2: Bezier Splines **(10 points)**

The Bernstein polynomials of degree *d* are:

$$b_i^{(d)}(t) = \binom{d}{i} t^i (1-t)^{d-i}, i = 0 \ldots d$$

(a) Show that these functions form a partition of unity on $t \in [0..1]$. (2 points)

(b) Determine the polar forms $B_i^{(2)}$ of the quadratic Bernstein polynomials $b_i^{(2)}$ (*calculation only*). (4 points)

(c) Make a drawing: Execute one subdivision step for the quadratic Bezier curve defined by the control points {(0,0), (5,5), (10,0)}. Sketch the control polygon. Subdivide the control polygon into two segments in the middle of the parameter domain ($t = 0.5$) using the de Castlejau algorithm. Now sketch the Bezier curve as well. What is the continuity with which the two newly created spline segments meet at parameter value $t = 0.5$? (*results only, brief explanation only for the last question*) (4 points)

Assignment #3: Implicit Functions **(25 points)**

(a) Construct a signed distance function for the unit square (in $\mathbb{R}^2$, centered at (0,0), side length 1, with sides parallel to the coordinate axes). The interior should be negative. (*no explanation*) (8 points)

(b) Provide a signed distance function for a unit sphere. The interior should be negative. (*no explanation*) (2 points)

(c) Provide an implicit function for the intersection of two unit spheres (diameter 1), centered at [-0.25, 0, 0], [0.25, 0, 0]. The interior should be negative. (*no explanation*) (7 points)

(d) Consider the following implicit function:

$$f(x,y)=\frac{3}{2}x^2+\frac{3}{2}y^2+xy-1$$

Is the zero level set of this function one of the following: a circle, an ellipse, a hyperbola, a parabola, a straight line, a point, or the empty set? (*please explain your answer*)

Hint: The vector [1,1] at some point turns out to be an eigenvector of something. (8 points)

Assignment #4: Piecewise Polynomial Interpolation **(15 points)**

(a) Assume we are given n data values $y_i \in \mathbb{R}$, $1 \leq i \leq n$, corresponding to data points (i, y_i). How can we find a function $f(x)$ that is piecewise quadratic on all unit intervals $(j,\ j+1)$, $1 \leq j \leq n-1$ and that interpolates the points (i, y_i) with C^1 continuity? Setup a linear system of equations that describes all functions f with this property. (*propose a solution, explain briefly, specify all unknowns and specify the conditions that form the linear system*) (10 points)
Hint: it is useful to first create a drawing of an example interpolation problem

(b) Does this system have a unique solution? If not, what could be done to make the solution unique? (*propose at least one solution, one or two sentences*) (5 points)

Assignment #5: B-Spline Subdivision **(15 points)**

Derive how uniform, quadratic B-spline functions $b^{(2)}(t)$ can be expressed by higher resolution dilated functions $b^{(2)}(2t+i)$, $i \in \mathbb{Z}$. Use the construction that is based on repeated convolution as presented in the lecture. Your result should be a formula for how to linearly combine the $b^{(2)}(2t+i)$ to form the function $b^{(2)}(\mathrm{t})$.

Hint: Compute the basis function $b^{(2)}$ by a repeated convolution of box functions $b^{(0)}$.
Replace the box functions by a sum of a pair of smaller box functions.

Below, you can find a few useful properties of convolutions:

Commutativity: $f(t)\otimes g(t)=g(t)\otimes f(t)$

Associativity: $f(t)\otimes\left(g(t)\otimes h(t)\right)=\left(f(t)\otimes g(t)\right)\otimes h(t)$

Distributivity: $f(t)\otimes\left(g(t)+h(t)\right)=f(t)\otimes g(t)+f(t)\otimes h(t)$

Scale and Shift: $f(at+b)\otimes g(at+c)=\frac{1}{a}(f\otimes g)(at+b+c)$

Assignment #6: Convex Hull **(10 points)**

Describe an algorithm that computes the convex hull of a set of n points in $\mathbb{R}^d$.

Input: A set X of points $\mathbf{x}_i \in \mathbb{R}^d$ in d-dimensional space ($1 \leq i \leq n$, $n > d$).
The points are in general position (no set of points with size $k+1$ lies in a k-dimensional affine subspace of $\mathbb{R}^d$).

Output: Minimal set of half-spaces $H(\mathbf{n}, c) = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{x} \cdot \mathbf{n} \geq c\}$ whose intersection describes the convex hull.

(*You do not need to specify computational details, such as how to exactly compute half spaces and their parameters. You can use either pseudo code or plain text for describing the algorithm*).