Geometric Modeling

Summer Semester 2012

Blossoming and Polar Forms

Piecewise Polynomial Splines Revisited

UNIVERSITÄT DES SAARLANDES

CLUSTER OF EXCELLENCE

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Today...

Topics:

- Introduction: Geometric Modeling
- Mathematical Background
- Interpolation & Approximation
- Splines
  - Polynomial Spline Curves
  - Blossoming and Polar Forms
  - Rational Splines
  - Spline Surfaces
- Meshes

Overview...

Topics:

- Blossoming and Polar Forms
  - The De Casteljau Algorithm
  - Polar Forms: Idea & Definition
  - Polynomial Splines Revisited
    - Bezier Splines
    - B-Splines

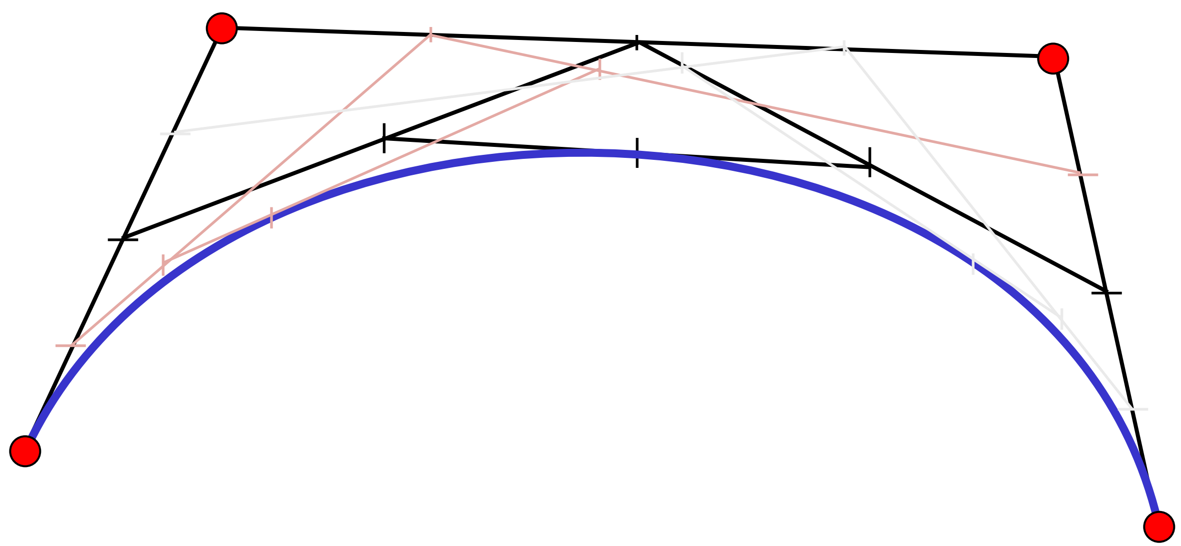
Geometric View:
The De Casteljau Algorithm

De Casteljau Algorithm

Idea of Bezier splines can be formulated differently:

- Geometric view
  - Repeated linear interpolation with common parameter t
  - Implicitly creates polynomial in t
  - Degree depends on number of cascaded interpolations
- Geometric interpretation: more intuitive
  - Properties of the Bezier spline can be interpreted geometrically
    - Derivatives
    - Operations (e.g. subdivision)
    - ...
- We will now look at the corresponding algorithm...

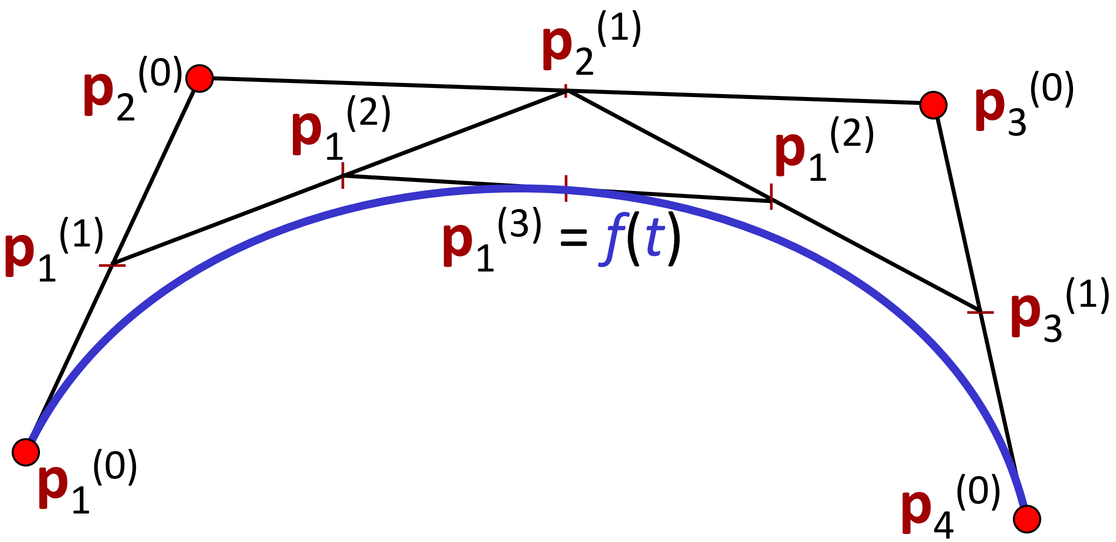
De Casteljau



De Casteljau Algorithm: Computes *f*(*t*) for given *t*

- Bisect control polygon in ratio *t* : (1 – *t*)
- Connect the new dots with lines (adjacent segments)
- Interpolate again with the same ratio
- Iterate, until only one point is left

De Casteljau



Algorithm:

for j = 0..d−1 **do**

    for i = 1..d−j **do**

        $\mathbf{p}_i^{(j)} = (1-t)\cdot\mathbf{p}_i^{(j-1)} + t\cdot\mathbf{p}_{i+1}^{(j-1)}$

    end for

end for

return $\mathbf{p}_1^{(d)}$

Properties

Properties:

- Yields same result as Bernstein basis formulation
- Iterated convex combinations of control points
  - Numerically more stable than monomial evaluation
  - Easy to see:
    - Affine invariant
    - Convex hull property
- Open questions: How to geometrically interpret
  - Derivatives
  - Operations (knot insertion, degree elevation etc.)
  - ...

Polar Forms & Blossoms:
Idea & Definition

Affine Combinations

First: A quick recap of "linear interpolation"

- Actually, the right name should be "affine interpolation"

Definition:

- An *affine combination* of *n* points $\in \mathbb{R}^d$ is given by:

$$\mathbf{p}_{\boldsymbol{\alpha}} = \sum_{i=1}^{n} \alpha_i \mathbf{p}_i \quad \text{with} \quad \sum_{i=1}^{n} \alpha_i = 1$$

- A function *f* is set to be *affine* in its parameter $\mathbf{x}_i$, if:

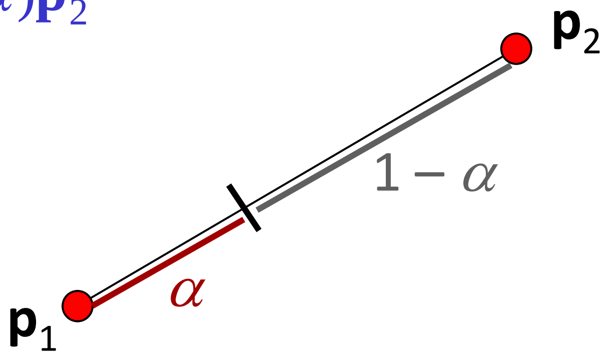
$$f\left(x_1, \ldots, \sum_{k=1}^{n} \alpha_i x_i^{(k)}, \ldots, \mathrm{x}_{\mathrm{m}}\right) = \sum_{k=1}^{n} \alpha_i f\left(x_1, \ldots, x_i^{(k)}, \ldots, \mathrm{x}_{\mathrm{m}}\right) \quad \text{for} \quad \sum_{i=1}^{n} \alpha_i = 1$$

Affine Combinations

Examples:

- Linear (affine) interpolation of 2 points:

$$\mathbf{p}_\alpha = \alpha\mathbf{p}_1 + (1-\alpha)\mathbf{p}_2$$

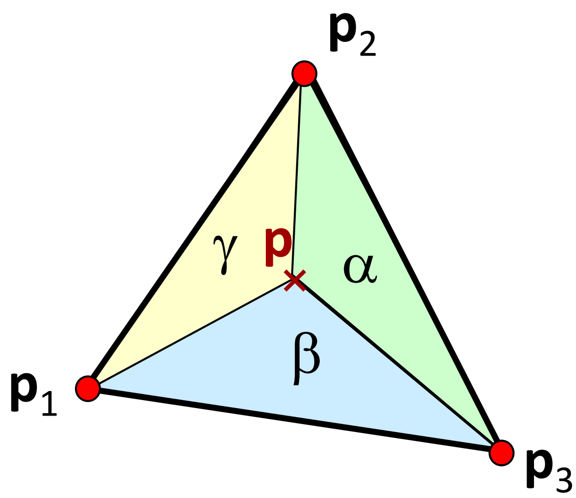


Affine Combinations

Examples:

- Barycentric combinations of 3 points ("barycentric coordinates")

  $\mathbf{p} = \alpha \mathbf{p}_1 + \beta \mathbf{p}_2 + \gamma \mathbf{p}_3, \text{with} : \alpha + \beta + \gamma = 1$



  Properties:

  $\gamma = 1 - \alpha - \beta$

  $$\alpha = \frac{area(\Delta(\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}))}{area(\Delta(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3))}, \beta = \frac{area(\Delta(\mathbf{p}_1, \mathbf{p}_3, \mathbf{p}))}{area(\Delta(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3))}, \gamma = \frac{area(\Delta(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}))}{area(\Delta(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3))}$$

  Transformation to barycentric coordinates is a linear map (heights in triangles).

Formalizing the Idea

Idea: Express (piecewise) polynomial curves as iterated linear (affine) interpolations

First try:

- A polynomial: $p(t) = at^3 + bt^2 + ct + d$
- Can be written as: $p(t) = a \cdot t \cdot t \cdot t + b \cdot t \cdot t + c \cdot t + d$
- Interpret each variable t a separate parameter:

  $$\mathbf{p}(t_1, t_2, t_3) = \mathbf{a} \cdot t_1 \cdot t_2 \cdot t_3 + \mathbf{b} \cdot t_1 \cdot t_2 + \mathbf{c} \cdot t_1 + \mathbf{d}$$

  - t_1 moves linearly in direction (**a** + **b** + **c**)
  - t_2 in direction (**a** + **b**)
  - t_3 in direction **a**
- Problems: fixed directions, many representations

Polar Forms

Improved solution: Polar Forms / Blossoms

A *polar form* or *blossom* f of a polynomial F of degree d is a function in d variables:

$$F: \quad \mathbb{R} \to \mathbb{R}$$
$$f: \quad \mathbb{R}^d \to \mathbb{R}$$

with the following properties:

- *Diagonality*: $f(t, t, \ldots, t) = F(t)$
- *Symmetry*: $f(t_1, t_2, \ldots, t_d) = f(t_{\pi(1)}, t_{\pi(2)}, \ldots, t_{\pi(d)})$
  for all permutations of indices π.
- *Multi-affine*: $\Sigma\alpha_k = 1$

$$\Rightarrow f(t_1, t_2, \ldots, \Sigma\alpha_k t_i^{(k)}, \ldots, t_d)$$
$$= \alpha_1 f(t_1, t_2, \ldots, t_i^{(1)}, \ldots, t_d) + \ldots + \alpha_n f(t_1, t_2, \ldots, t_i^{(n)}, \ldots, t_d)$$

Polar Forms

Based on the same idea as on slide 9:

- Model polynomial as multi-affine function (multi-affinity property)
- Plugging in a common parameter to obtain the original polynomial
- *New:* Symmetry property – makes the solution unique
  - There is exactly one polar form for each polynomial
  - This standardization makes different polars "compatible", we can compare them with each other
  - We will see how this works in a few slides...

Properties

Properties of polar forms:

- The mapping from polynomials to their polar forms is one-to-one:
  - For each polar form $f(t_1,t_2,\ldots,t_n)$,
    a unique polynomial $F(t,t,\ldots,t)$ exists
  - For each polynomial F,
    a unique polar form $f(t_1,t_2,\ldots,t_n)$ exists

Properties

Properties of polar forms:

- Polar forms of monomials:
  - Degree 0: $1 \to 1$
  - Degree 1: $1 \to 1,\ t \to t$
  - Degree 2: $1 \to 1,\ t \to \dfrac{t_1 + t_2}{2},\ t \to t_1 t_2$
  - Degree 3: $1 \to 1,\quad t^2 \to \dfrac{t_1 t_2 + t_2 t_3 + t_1 t_3}{3},$

    $t \to \dfrac{t_1 + t_2 + t_3}{3},\quad t^3 \to t_1 t_2 t_3$

Properties

Properties of polar forms:

- Polar forms of monomials:
  - Degree 0: $f = c_0$
  - Degree 1: $f(t_1) = c_0 + c_1 t_1$
  - Degree 2: $f(t_1, t_2) = c_0 + c_1 \dfrac{t_1 + t_2}{2} + c_2 t_1 t_2$
  - Degree 3: $f(t_1, t_2, t_3) = c_0 + c_1 \dfrac{t_1 + t_2 + t_3}{3} + c_2 \dfrac{t_1 t_2 + t_2 t_3 + t_1 t_3}{3} + c_3 t_1 t_2 t_3$

Properties

General Case:

- $f(t_1,\ldots,t_n) = \sum_{i=0}^{n} c_i \binom{n}{i}^{-1} \sum_{\substack{S\subseteq\{1..n\},\\ |S|=i}} \prod_{j\in S} t_i$
- The c_i are the monomial coefficients.
- Idea: Use all possible subsets of t_i to make it symmetric.
- This solution is unique.
- Without the symmetry property, there would be a large number of solutions.

Generalizations

Blossoms for polynomial curves (points as output):

- Polar form of a polynomial curve of degree d:

  $\mathbf{F}: \quad \mathbb{R} \rightarrow \mathbb{R}^{n}$

  new

  $\mathbf{f}: \quad \mathbb{R}^{d} \rightarrow \mathbb{R}^{n}$

- Required Properties:
  - Diagonality: $\mathbf{f}(t, t, \ldots, t) = \mathbf{F}(t)$
  - Symmetry: $\mathbf{f}(t_1, t_2, \ldots, t_d) = \mathbf{f}(t_{\pi(1)}, t_{\pi(2)}, \ldots, t_{\pi(d)})$
    for all permutations of indices π.
  - Multi-affine: $\Sigma\alpha_k = 1$

    $\Rightarrow \mathbf{f}(t_1, t_2, \ldots, \Sigma\alpha_k t_i^{(k)}, \ldots, t_d)$

    $= \alpha_1 \mathbf{f}(t_1, t_2, \ldots, t_i^{(1)}, \ldots, t_d) + \ldots + \alpha_n \mathbf{f}(t_1, t_2, \ldots, t_i^{(n)}, \ldots, t_d)$

Generalizations

Blossoms with points as arguments:

- Polar form degree d with points as input und output:

  $\mathbf{F}: \quad \mathbb{R}^{\boxed{m}} \to \mathbb{R}^n$ new

  $\mathbf{f}: \quad \mathbb{R}^{d \times \boxed{m}} \to \mathbb{R}^n$

- Required Properties:
  - Diagonality: $\mathbf{f}(\mathbf{t}, \mathbf{t}, \ldots, \mathbf{t}) = \mathbf{F}(\mathbf{t})$
  - Symmetry: $\mathbf{f}(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_d) = \mathbf{f}(\mathbf{t}_{\pi(1)}, \mathbf{t}_{\pi(2)}, \ldots, \mathbf{t}_{\pi(d)})$
    for all permutations of indices π.
  - Multi-affine: $\Sigma\alpha_k = 1$

    $\Rightarrow \mathbf{f}(\mathbf{t}_1, \mathbf{t}_2, \ldots, \Sigma\alpha_k \mathbf{t}_i^{(k)}, \ldots, \mathbf{t}_d)$

    $= \alpha_1 \mathbf{f}(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_i^{(1)}, \ldots, \mathbf{t}_d) + \ldots + \alpha_n \mathbf{f}(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_i^{(n)}, \ldots, \mathbf{t}_d)$

Generalizations

Vector arguments

- We will have to distinguish between *points* and *vectors* (differences of points)
- Use "hat" notation $\hat{\mathbf{v}} = \mathbf{p} - \mathbf{q}$ to denote vectors (differences of points)
- Also defined in the one dimensional case (vectors in $\mathbb{R}$)
- One vector: $\hat{1} = 1 - 0$, $\hat{\mathbf{1}} = [1,...,1]^T - \mathbf{0}$
- Define shorthand notation (recursive):

$$f(\underbrace{t_1,...,t_{n-k}}_{n-k},\underbrace{\hat{v}_1,...,\hat{v}_k}_{k}) := f(\underbrace{t_1,...,t_{n-k}}_{n-k},p_1,\underbrace{\hat{v}_2,...,\hat{v}_k}_{k-1}) - f(\underbrace{t_1,...,t_{n-k}}_{n-k},q_1,\underbrace{\hat{v}_2,...,\hat{v}_k}_{k-1})$$

Properties

Derivatives of blossoms:

- $$f(t_1,\ldots,t_n) = \sum_{i=0}^{n} c_i \binom{n}{i}^{-1} \sum_{\substack{S \subseteq \{1..n\},\\ |S|=i}} \prod_{j \in S} t_i$$
- The c_i are related to the derivatives at t = 0.
- Hence: $$c_i = \frac{\frac{d^k}{dt^k} F(0)}{k!} = \binom{n}{k} f(\underbrace{0,\ldots,0}_{n-k},\underbrace{\hat{1},\ldots,\hat{1}}_{k})$$
- In general: $$\frac{d^k}{dt^k} F(t) = \frac{n!}{(n-k)!} f(\underbrace{t,\ldots,t}_{n-k},\underbrace{\hat{1},\ldots,\hat{1}}_{k})$$

Example

Example:

$$f(t_1,t_2,t_3)=c_0+c_1\frac{t_1+t_2+t_3}{3}+c_2\frac{t_1t_2+t_2t_3+t_1t_3}{3}+c_3t_1t_2t_3$$

$$f'(t)=\frac{3!}{2!}\left[\left(c_0+c_1\frac{1+t+t}{3}+c_2\frac{1t+tt+1t}{3}+c_31tt\right)-\left(c_0+c_1\frac{0+t+t}{3}+c_2\frac{tt}{3}\right)\right]$$

$$=3\left(c_1\frac{1}{3}+c_2\frac{2t}{3}+c_3tt\right)$$

$$=3c_3t^2+2tc_2+c_1$$

Continuity Condition

Theorem: Continuity condition for polynomials

The following statements are equivalent:

- F and G are C^k-continuous at t
- $\forall t_1, \ldots, t_k : f(t, \ldots, t, t_1, \ldots, t_k) = g(t, \ldots, t, t_1, \ldots, t_k)$
- $f(t, \ldots, t, \underbrace{\hat{1}, \ldots, \hat{1}}_{k\text{-times}}) = g(t, \ldots, t, \underbrace{\hat{1}, \ldots, \hat{1}}_{k\text{-times}})$ *)

*) 2 $\Leftrightarrow$ 3: $$\begin{aligned} f(t, \ldots, t, t_1) &= f(t, \ldots, t, (t_1 - 0)) \\ &= t_1 f(t, \ldots, t, 1) - f(t, \ldots, t, 0) \\ &= t_1 f(t, \ldots, t, \hat{1}) \end{aligned}$$

Continuity Condition

Examples:

- $\forall t_1, t_2, t_3$: $f(t_1, t_2, t_3) = g(t_1, t_2, t_3) \Rightarrow$ same curve
- $\forall t_1, t_2$: $f(t_1, t_2, t) = g(t_1, t_2, t) \Rightarrow C^2$ at t
- $\forall t_1$: $f(t_1, t, t) = g(t_1, t, t) \Rightarrow C^1$ at t
- $f(t, t, t) = g(t, t, t) \Rightarrow C^0$ at t

Raising the Degree

Raising the degree of a blossom:

- Can we directly construct a polar form with degree elevated by one from a lower degree one, without changing the polynomial?
- [other than transforming to monomials, adding $0 \cdot t^{d+1}$, and transforming back?]

Solution:

- Given: $f(t_1,..,t_d)$
- We obtain: $f^{(+1)}(t_1,..,t_{d+1}) = \frac{1}{d+1}\sum_{i=1}^{d+1} f(t_1,..,t_{i-1},t_{i+1},..,t_{d+1})$

leave out t_i

Raising the Degree

Proof:

$$\forall t: f^{(+1)}(t,..,t) = \frac{1}{d+1}\sum_{i=1}^{d+1} f(t_1,..,t_{i-1},t_{i+1},..,t_{d+1})\Big|_{t_1 = .. = t_{d+1} = t}$$

$$= \frac{1}{d+1}\sum_{i=1}^{d+1} f(t,....,t)$$

$$= f(t,...,t)$$

$$\Rightarrow F^{(+1)}(t) = F(t)$$

Polars and Control Points

Interpretation (Examples):

- Multi-variate function: $\mathbf{f}(t_1, t_2, t_3)$
  - Describes a point depending on three parameters
  - Where $\mathbf{f}(t_1, t_2, t_3)$ moves for changing (t_1, t_2, t_3) depends on f (think of storing monomial coefficients inside)
- Polynomial value: $\mathbf{f}(1.5, 1.5, 1.5)$
  - One value of the polynomial curve: $\mathbf{F}(1.5)$
- Off-curve points: $\mathbf{f}(1, 2, 3)$
  - Evaluate points not necessarily on the polynomial curve
  - Question: What meaning do various off-curve points have?
  - We will use of-curve points as control points

Polars and Control Points

Interpretation (Examples):

- Specifying $\mathbf{f}(t_1, t_2, t_3)$:
  - Assume, f is not know yet
  - We want to determine a polar (i.e. a polynomial)
- On curve points:
  $\{\mathbf{f}(0,0,0) = \mathbf{x}_0, \mathbf{f}(1,1,1) = \mathbf{x}_1, \mathbf{f}(2,2,2) = \mathbf{x}_2, \mathbf{f}(3,3,3) = \mathbf{x}_3\}$
  - Degree d polynomial has d+1 degrees of freedom
  - We know already how to do polynomial interpolation
- Off-curve points:
  $\{\mathbf{f}(1,1,1) = \mathbf{x}_{111}, \mathbf{f}(1,2,3) = \mathbf{x}_{123}, \mathbf{f}(2,3,4) = \mathbf{x}_{234}, \mathbf{f}(3,3,3) = \mathbf{x}_{333}\}$
  - We can also use off-curve points to specify the polar/polynomial
  - This is the main motivation for the whole formalism

Polynomial Splines Revisited:

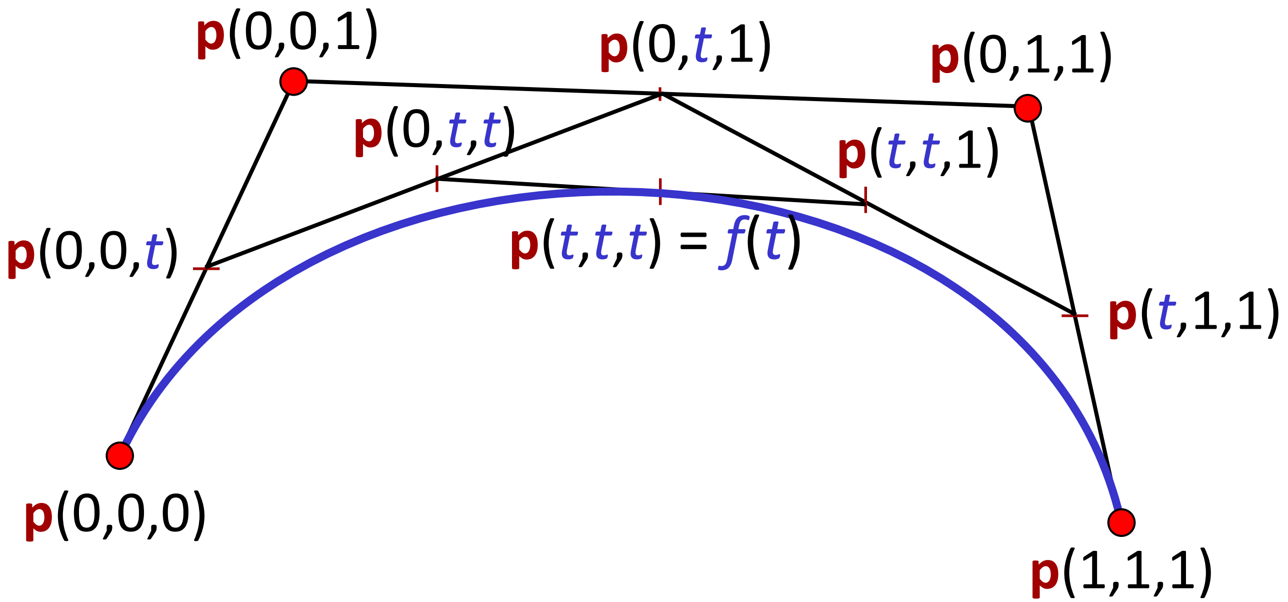
Bezier Splines

De Casteljau Algorithm

The de Casteljau algorithm is simple to state with blossoms:

- We just have to exchange the labels
- Then use the multi-affinity property in order to compute intermediate points
- With this view, we can easily show that the de Casteljau algorithm is equivalent to the formulation based on Bernstein polynomials

De Casteljau



Bezier control points: **p**(0,0,0), **p**(0,0,1), **p**(0,1,1), **p**(1,1,1)

Analysis

Transforming a polar to the Bernstein basis:

$$
\begin{aligned}
\mathbf{f}(t,\ldots,t) &= (1-t)\mathbf{f}(t,..,t,0) + t\mathbf{f}(t,..,t,1) \\
&= (1-t)\left[(1-t)\mathbf{f}(t,..,t,0,0) + t\mathbf{f}(t,..,t,0,1)\right] + t\left[(1-t)\mathbf{f}(t,..,t,1,0) + t\mathbf{f}(t,..,t,1,1)\right] \\
&= (1-t)^2\mathbf{f}(t,..,t,0,0) + 2t(1-t)\mathbf{f}(t,..,t,0,1) - t^2\mathbf{f}(t,..,t,1,1) \\
&= \ldots \\
&= \sum_{i=0}^{n}\binom{n}{i} t^i (1-t)^{n-i}\mathbf{f}(\underbrace{0,..,0}_{n-i},\underbrace{1,..,1}_{i})
\end{aligned}
$$

Analysis

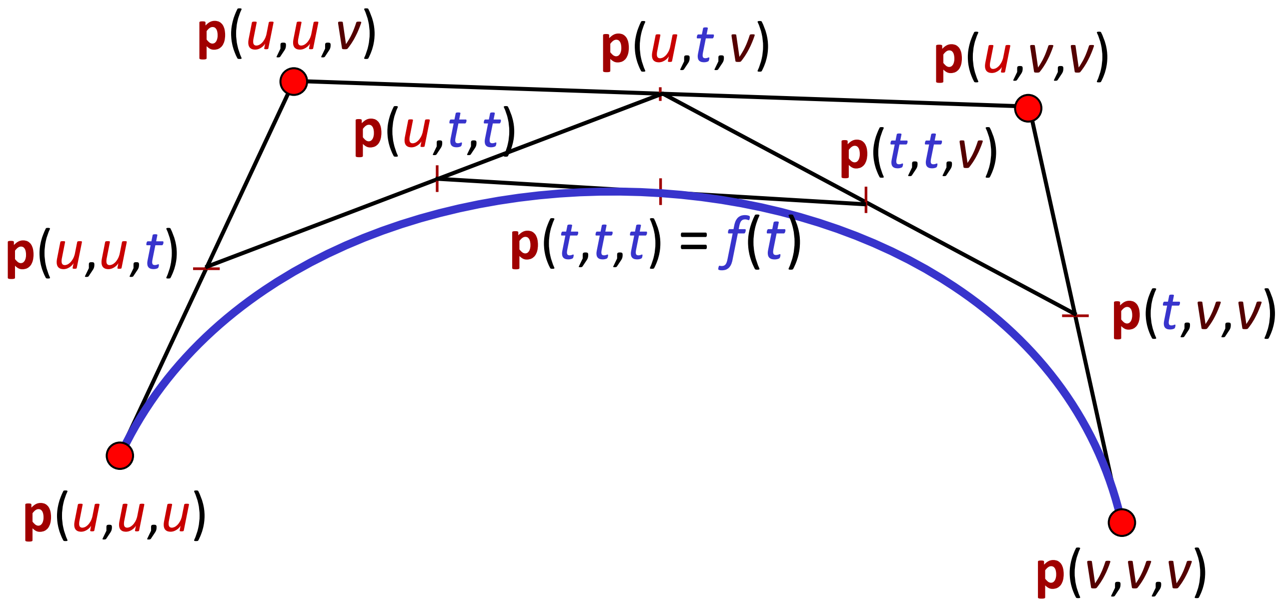
De Castlejau Algorithm: Performs this in reverse order

- Bezier points: $\mathbf{p}_i^{(0)}(t) = \mathbf{f}(\underbrace{0,..,0}_{d-i},\underbrace{1,..,1}_{i},)$
- Intermediate points: $\mathbf{p}_i^{(j)}(t) = \mathbf{f}(\underbrace{0,..,0}_{d-i-j},\underbrace{1,..,1}_{i},\underbrace{t,..,t}_{j})$
- Recursive computation:

$$\begin{aligned}
\mathbf{p}_i^{(j)}(t) &= \mathbf{f}(\underbrace{0,..,0}_{d-i-j},\underbrace{t,..,t}_{j},\underbrace{1,..,1}_{i}) \\
&= (1-t)\mathbf{f}(\underbrace{0,..,0}_{d-i-j+1},\underbrace{t,..,t}_{j-1},\underbrace{1,..,1}_{i}) + t\mathbf{f}(\underbrace{0,..,0}_{d-i-j},\underbrace{t,..,t}_{j-1},\underbrace{1,..,1}_{i+1}) \\
&= (1-t)\mathbf{p}_i^{(j-1)}(t) + t\mathbf{p}_{i+1}^{(j-1)}(t)
\end{aligned}$$

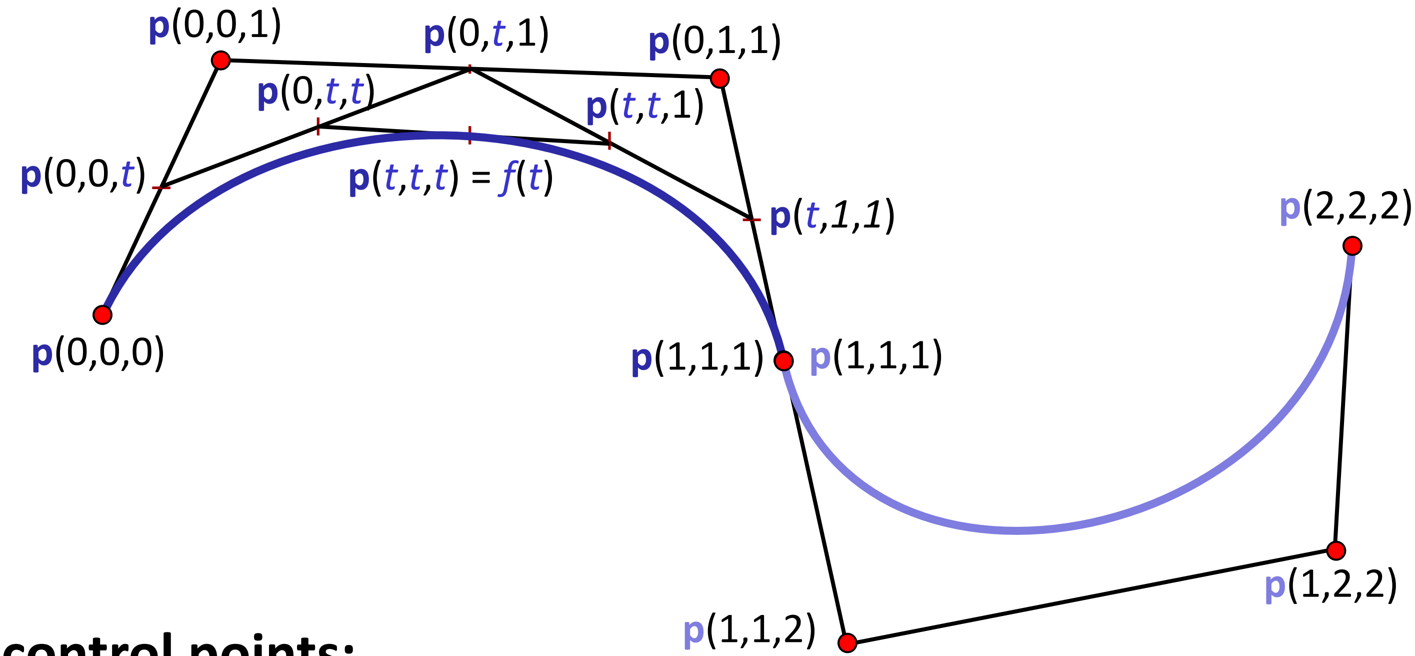
Consequence: Bernstein / de Casteljau lead to the same result

Generalized Parameter Intervals



Bezier control points: $\mathbf{p}(u,u,u)$, $\mathbf{p}(u,u,v)$, $\mathbf{p}(u,v,v)$, $\mathbf{p}(v,v,v)$

Multiple Segments



Bezier control points:

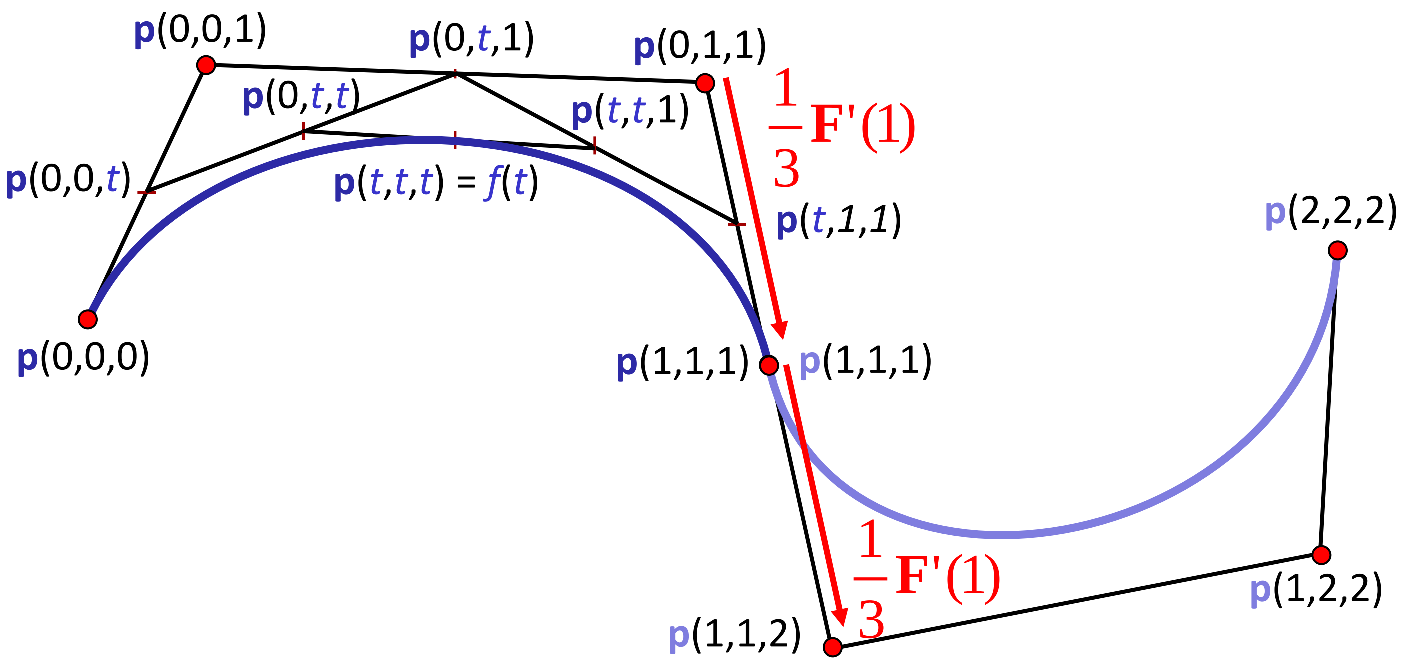
$\mathbf{p}(0,0,0)$, $\mathbf{p}(0,0,1)$, $\mathbf{p}(0,1,1)$, $\mathbf{p}(1,1,1) = \mathbf{p}(1,1,1)$, $\mathbf{p}(1,1,2)$, $\mathbf{p}(1,2,2)$, $\mathbf{p}(2,2,2)$

Two Curve Segments:

$\{\mathbf{p}(0,0,0), \mathbf{p}(0,0,1), \mathbf{p}(0,1,1), \mathbf{p}(1,1,1)\}$, $\{\mathbf{p}(1,1,1), \mathbf{p}(1,1,2), \mathbf{p}(1,2,2), \mathbf{p}(2,2,2)\}$

Remark: no interpolation between different segments
(e.g.: combination of $\mathbf{p}(0,1,1)$ and $\mathbf{p}(2,1,1)$ is not defined)

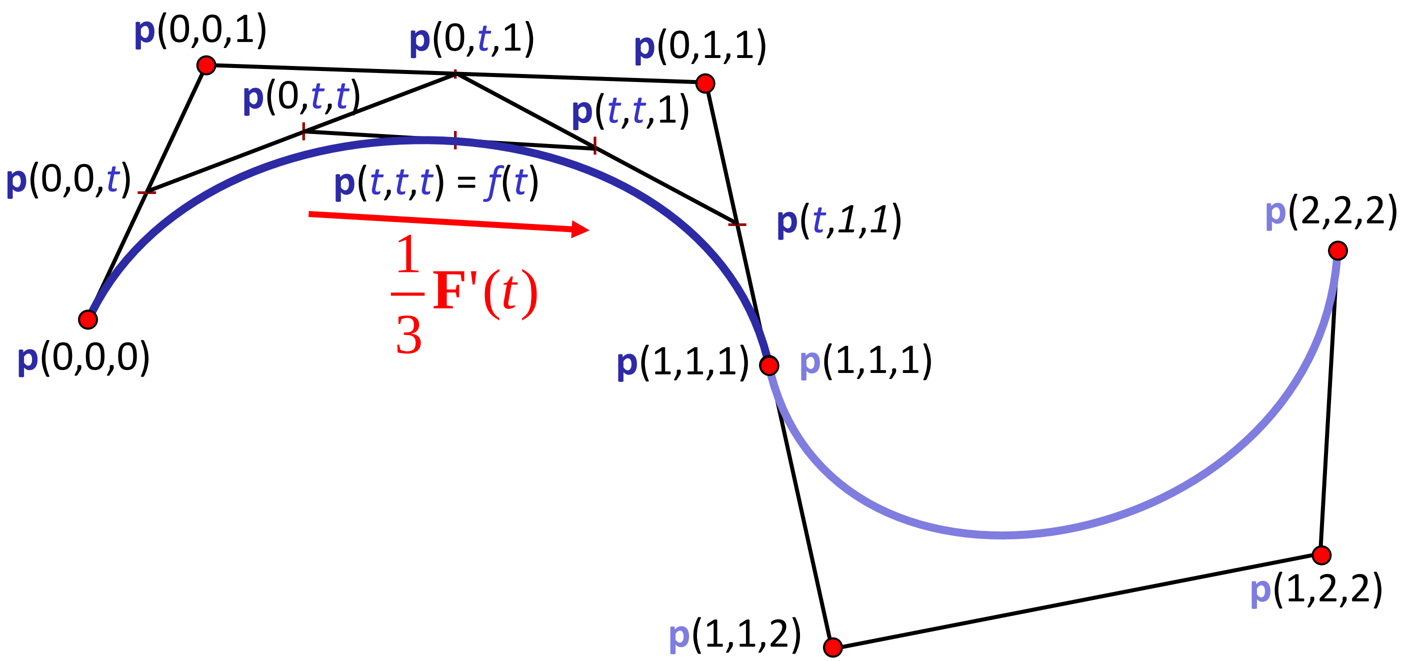
More Observations



Derivatives:

- $\frac{d}{dt}F(t) = d\ f(\underbrace{t,..,t}_{d-1},\hat{1}) = d\big(f(t,..,t,t+1) - f(t,..,t,t)\big)$ (degree d)
- C^1 Continuity condition follows

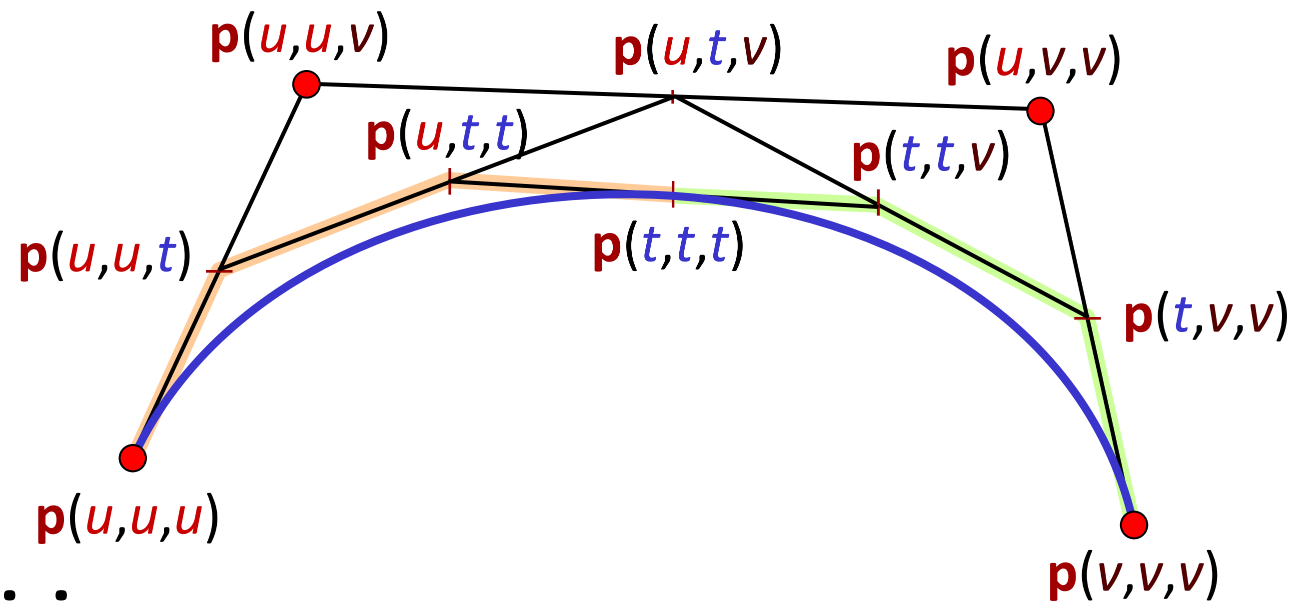
More Observations



Derivatives:

- De Casteljau Algorithm computes tangent vectors at any point as a byproduct
- Proportional to last line segment that is bisected

More Observations



Subdivision:

- After each de Casteljau step, we obtain two new control polygons left and right of $f(t)$ describing the same curve.
- We can divide a segment into two.
- Recursive subdivision can be used for rendering

Observations

Remark: The de Casteljau algorithm for computing

- Derivatives
  - at endpoints
  - at inner points t
- Subdivisions

hold for Bezier curves of arbitrary degree $d \geq 1$.

(General degree derivatives: $1/d\ \mathbf{F}'(t)$)

More Bezier Curve Properties...

General degree elevation:

- Increase the degree of a Bezier curve segment by one.
- What are the new control points?

Polar forms:

- Old curve: $\mathbf{b}(t_1,..,t_d)$
- New curve: $\mathbf{b}^{(+1)}(t_1,..,t_{d+1}) = \frac{1}{d+1}\sum_{i=1}^{d+1}\mathbf{b}(t_1,..,t_{i-1},t_{i+1},..,t_{d+1})$

leave out t_i

Degree Elevation

$$\mathbf{b}^{(+1)}(0,...,0) = \frac{1}{d+1}\sum_{i=1}^{d+1}\mathbf{b}(0,...,0) = \mathbf{b}(0,...,0)$$

$$\mathbf{b}^{(+1)}(1,0..,0) = \frac{1}{d+1}\mathbf{b}(0,...,0) + \frac{d}{d+1}\mathbf{b}(1,0,...,0)$$

$$\mathbf{b}^{(+1)}(1,1,0..,0) = \frac{2}{d+1}\mathbf{b}(1,0,...,0) + \frac{d-1}{d+1}\mathbf{b}(1,1,0,...,0)$$

$$\mathbf{b}^{(+1)}(1,1,1..,1,0) = \frac{d}{d+1}\mathbf{b}(1,..1,0) + \frac{1}{d+1}\mathbf{b}(1,1,1,...,1)$$

$$\mathbf{b}^{(+1)}(1,...,1) = \mathbf{b}(1,...,1)$$

Degree Elevation

Result: new control points

$$\mathbf{p}_i^{(+1)} = \frac{i}{n+1}\mathbf{p}_{i-1} + \left(1 - \frac{i}{n+1}\right)\mathbf{p}_i, \quad i = 0,\ldots,n+1 \ \text{(zero points if out of range)}$$

Repeated degree elevation:

$$\mathbf{p}_i^{(+k)} = \sum_{j=1}^{d} \mathbf{p}_j \binom{d}{j} \frac{\binom{k}{i-j}}{\binom{d+k}{j}} \quad \text{(proof by induction)}$$

Repeating degree elevation lets the control point converge to the Bezier curve in the limit (proof using Stirling's formula).

Variation Diminishing Property

Settings:

- Given an original curve $C^{(original)} \subseteq \mathbb{R}^3$
- Given a second curve $\mathbb{R}^3 \supseteq C^{(derived)} = f(C^{(original)})$ that is derived from the original by some mapping (algorithm) f.
- If for any arbitrary plane P, $C^{(derived)}$ does not have more intersections*) with P than $C^{(original)}$, the mapping f is called *variation diminishing*.
- Formally: $\forall$ planes P: $\#(C^{(derived)} \cap P) \leq \#(C^{(original)} \cap P)$

Mappings of interest:

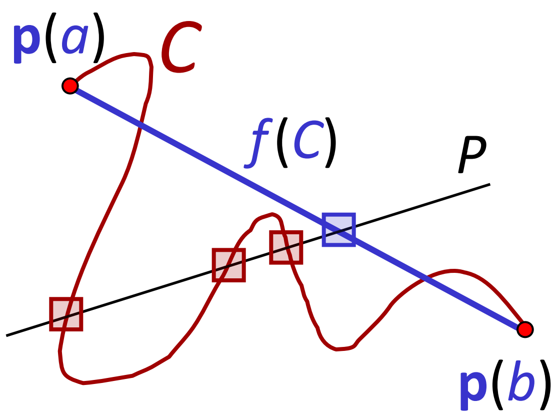
- Mapping from a control polygon to the curve segment

*) Intersection = crossing the plane

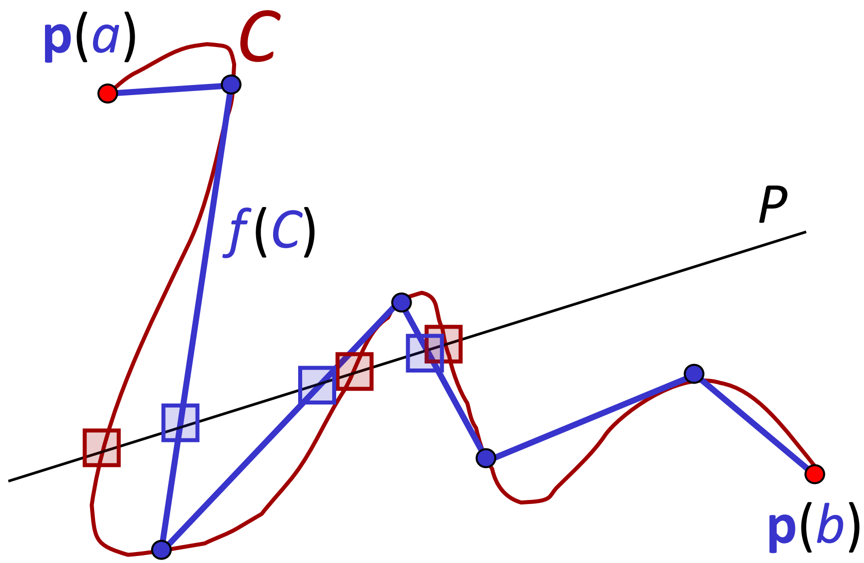
Easy example

Very simple mapping:

- Given a continuous curve $C = \mathbf{p}([a, b])$
- The mapping f creates a linear approximation:
  the line through $\mathbf{p}(a)$ and $\mathbf{p}(b)$.
- This mapping is variation diminishing
  - The line segment intersects the plane at most once
  - If this is the case, the original curve must have at least one intersection, too, because of continuity



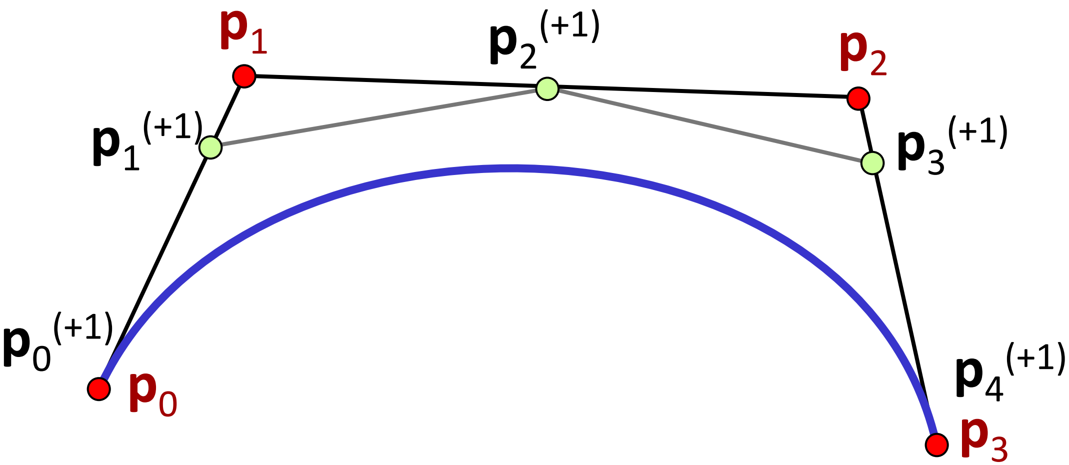
Another Example



Piecewise linear approximation:

- Obviously, a piecewise linear approximation of a curve (approximating with a polygon) is variation diminishing, too.

Bezier Curves



Degree Elevation:

$$\mathbf{p}_i^{(+1)} = \frac{i}{n+1}\mathbf{p}_{i-1} + \left(1 - \frac{i}{n+1}\right)\mathbf{p}_i$$

Bezier curves are variation diminishing:

- The Bezier curve is the limit of the control point sequence for infinite degree elevation.
- Every degree elevation step produces a piecewise linear approximation of the original control polygon, which is variation diminishing.

Bezier Curves

Consequence:

- A Bezier curve does not intersect any plane more frequently than its control polygon.
- Therefore, it cannot have too weird oscillations.
- However
  - The convergence of degree elevation is very slow
  - Not useful as evaluation technique in practice
  - One can approximate arbitrary smooth functions in a convergent way using the Bernstein basis (Weierstraß approximation theorem). But:
    - The convergence rate is not satisfactory in practice.
    - We do not have local control.

Polynomial Splines Revisited:

B-Splines

B-Spline Curves

(General) B-Spline Curves:

- Given:
  - A degree d (constant for the whole curve)
  - A knot sequence (t_0, t_1, ..., t_n) with $t_0 \leq t_1 \leq ... \leq t_n$
  - Control points $\mathbf{p}_0$, $\mathbf{p}_1$, ..., $\mathbf{p}_m$ ($m = n - d + 1$)
- With this information, we want to construct the spline curve within $t = [t_d, \ldots, t_{n-d}]$
- Each polynomial segment $\mathbf{f}_i(t)$ is defined by the $d + 1$ control points $\mathbf{p}_i, \ldots, \mathbf{p}_{i+d}$.
- Within each such segment / control point set, we can apply linear interpolation based on blossoms to compute points on the curve.

De Boor Algorithm

Blossoming version of the de Boor algorithm:

- The de Boor algorithm will appear as a generalization of the de Casteljau algorithm
  - similar structure
  - de Casteljau as a special case (special knot sequence)

De Boor Algorithm

Blossoming definition of B-Splines: very simple

- The control point sequence is given as:

  $\mathbf{p}_0 = \mathbf{p}(t_0,\ldots,t_{d-1})$

  $\ldots$

  $\mathbf{p}_i = \mathbf{p}(t_i,\ldots,t_{i+d-1})$

  $\ldots$

  $\mathbf{p}_m = \mathbf{p}(t_{n-d+1},\ldots,t_n)$
- This means: we just use consecutive knot values as blossom arguments
- Then, we proceed as before
- Same computational scheme as de Casteljau, but different weights (because of the different knot values)

De Boor Algorithm

For simplicity, we will first look at a single B-spline segment:

- Knots (t_1, t_1, ..., t_{2d})
- Control points $\mathbf{p}_1$, $\mathbf{p}_1$, ..., $\mathbf{p}_{d+1}$
- B-spline segment within [t_d...t_{d+1}]

quadratic,
interval [1,2]

f: 0,1,2,3,4,5,6,7,8

Interpolation rule:

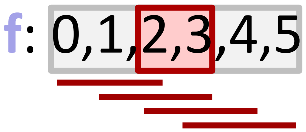
- Given blossoms f(a, t_1, ..., $t_{d\text{-}1}$), f(b, t_1, ..., $t_{d\text{-}1}$)
- We obtain f(t, t_1, ..., $t_{d\text{-}1}$) as:

$$f(t, t_1, .., t_{d-1}) = \frac{b-t}{b-a} f(a, t_1, .., t_{d-1}) + \left[1 - \frac{b-t}{b-a}\right] f(b, t_1, .., t_{d-1})$$

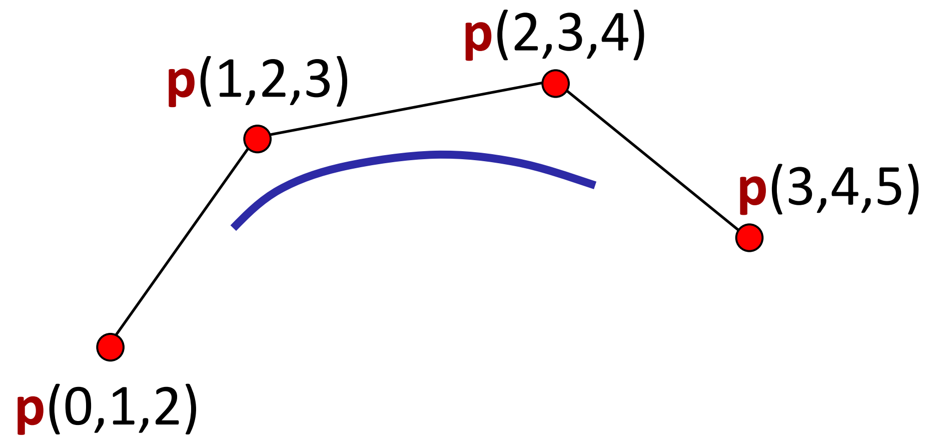
** updated **

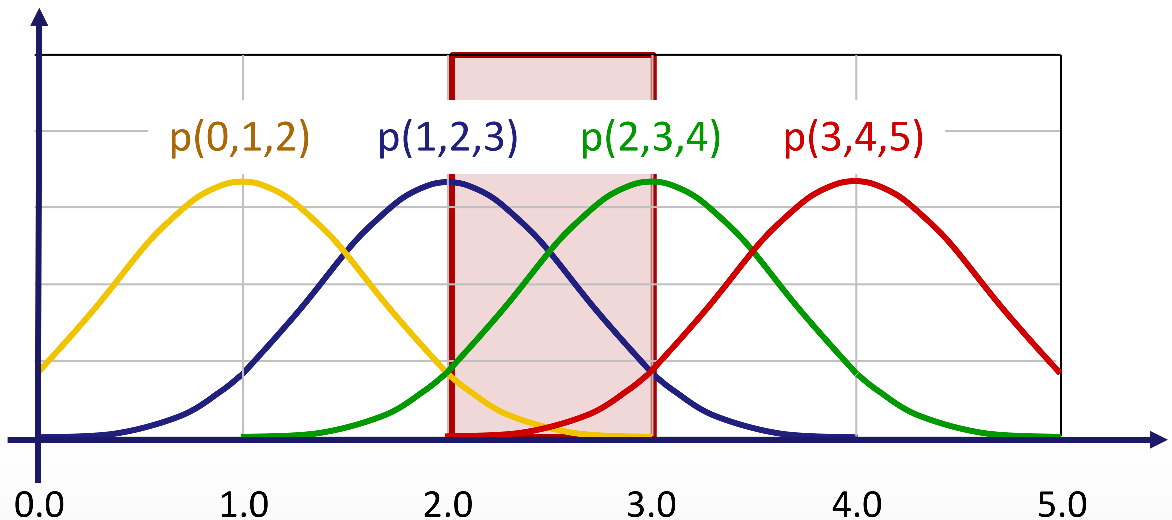
De Boor Algorithm

cubic,
interval [2,3]



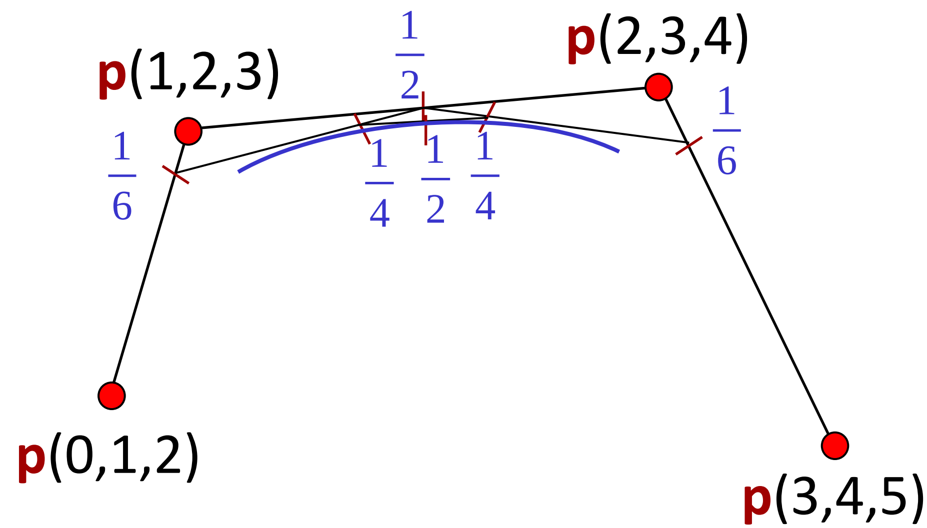
2d knots
("middle" interval
always exists)





** updated **

De Boor Algorithm



$$\frac{3-t}{3}\,\mathbf{p}(0,1,2) + \left[1-\frac{3-t}{3}\right]\mathbf{p}(3,1,2)$$

$$\frac{4-t}{3}\,\mathbf{p}(1,2,3) + \left[1-\frac{4-t}{3}\right]\mathbf{p}(4,2,3)$$

$$\frac{5-t}{3}\,\mathbf{p}(2,3,4) + \left[1-\frac{5-t}{3}\right]\mathbf{p}(5,3,4)$$

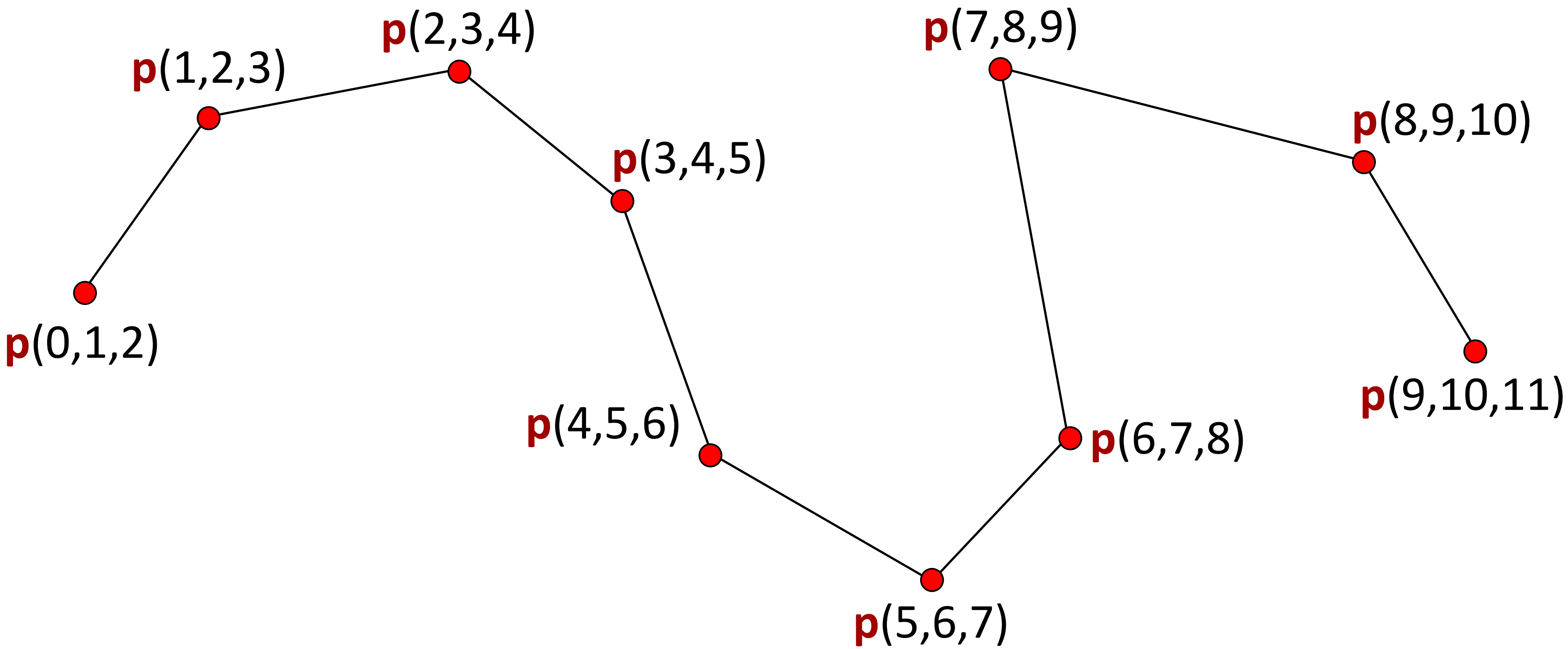
$$\frac{3-t}{2}\,\mathbf{p}(t,1,2) + \left[1-\frac{3-t}{2}\right]\mathbf{p}(t,3,2)$$

$$\frac{4-t}{2}\,\mathbf{p}(t,3,2) + \left[1-\frac{4-t}{2}\right]\mathbf{p}(t,3,4)$$

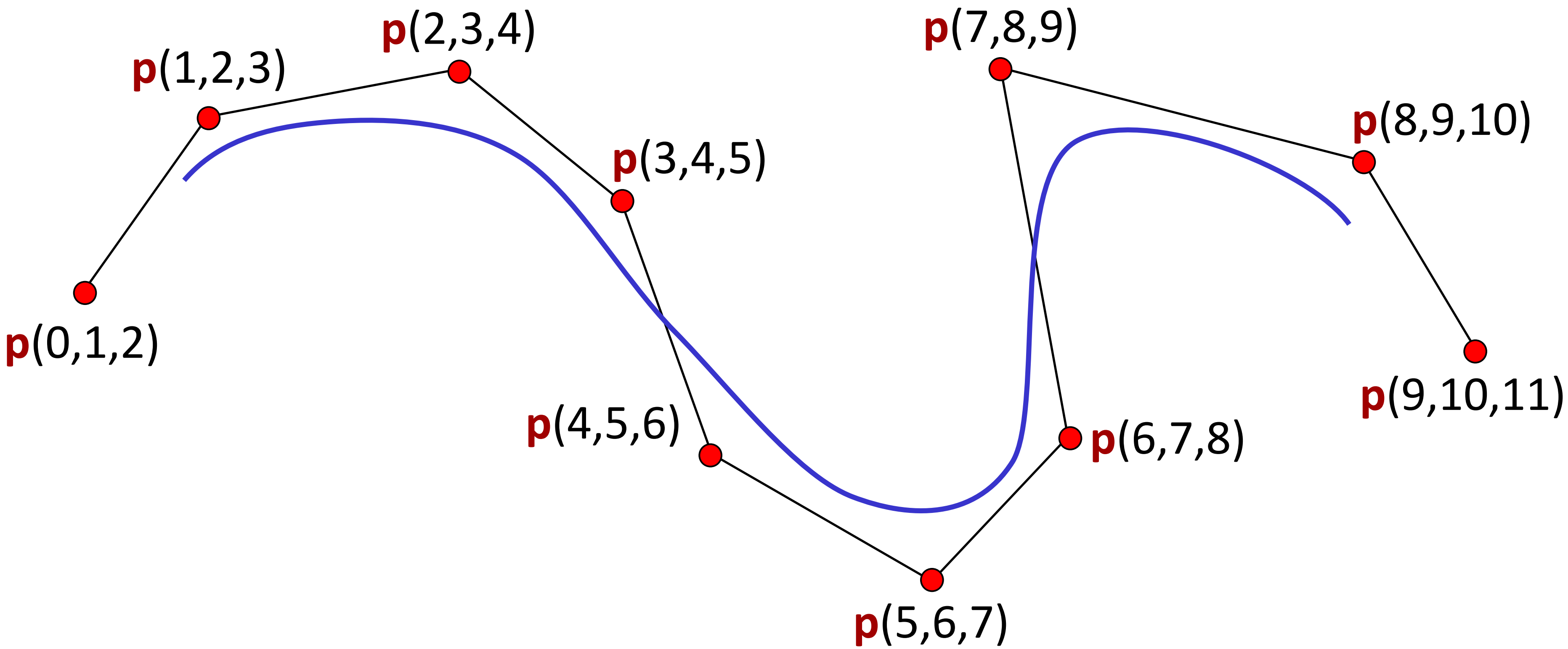
$$\frac{3-t}{1}\,\mathbf{p}(t,t,2) + \left[1-\frac{3-t}{1}\right]\mathbf{p}(t,t,3)$$

$$\mathbf{p}(t,t,t)$$

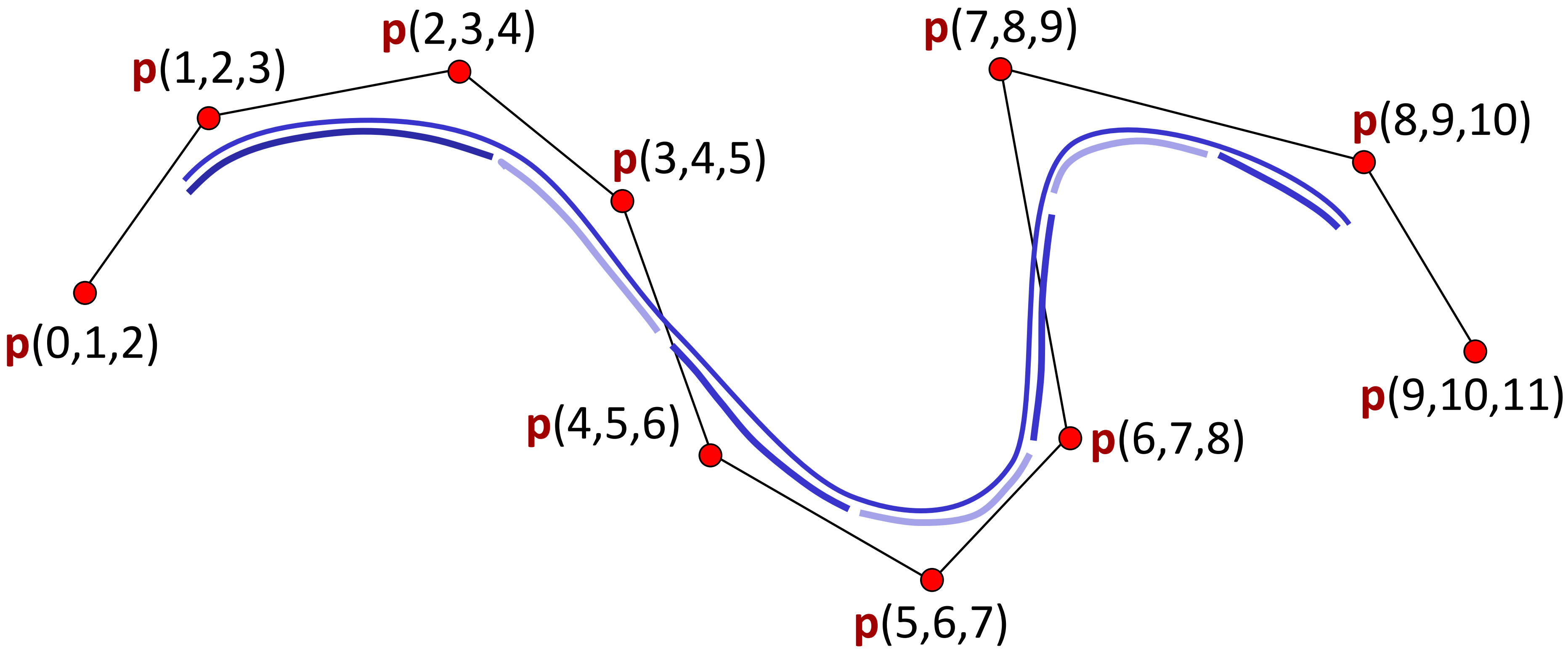
Example: General Case



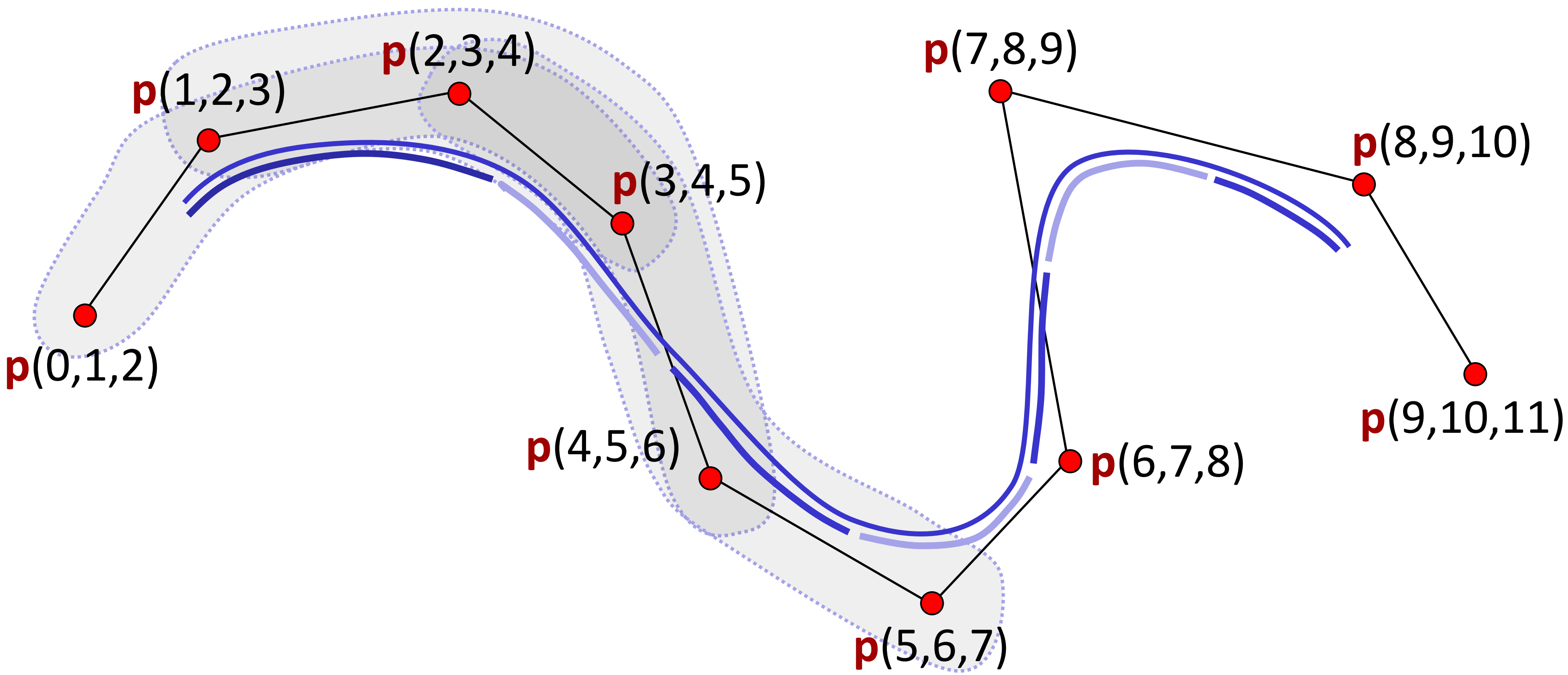
Example: General Case



Example: General Case



Example: General Case



De Boor Algorithm

De Boor Algorithm:

- In order to evaluate $f(t)$ for a $t \in [t_{i-1},...,t_i)$
- Compute:

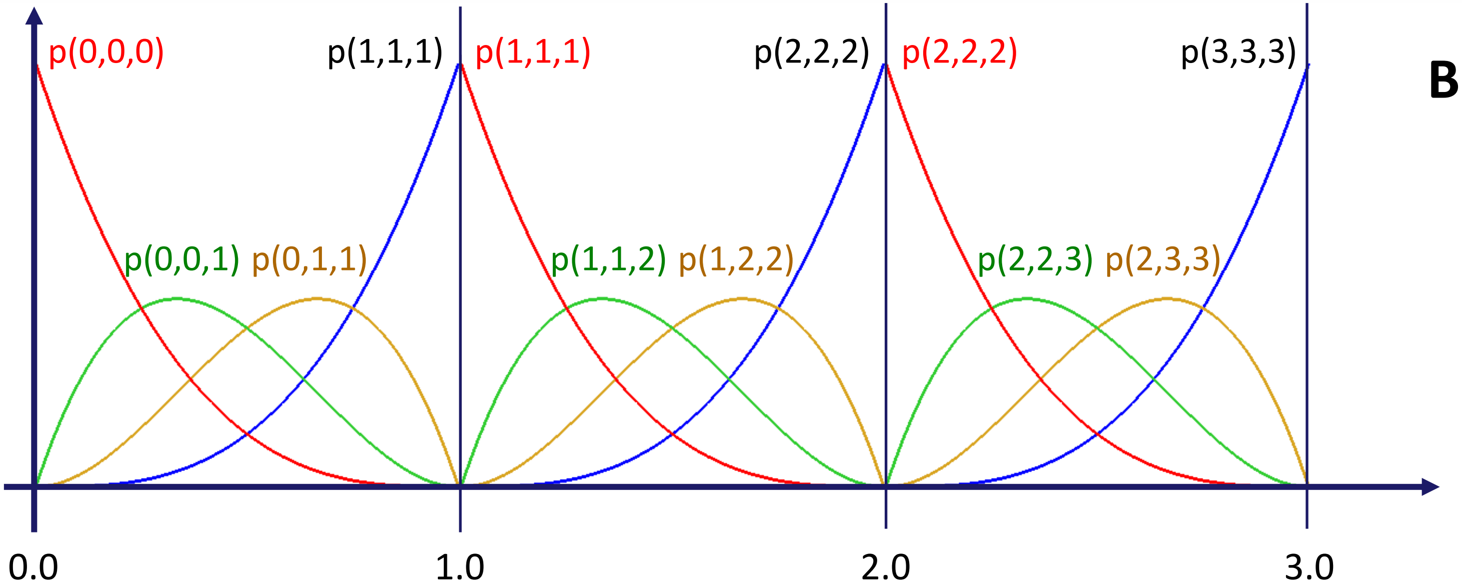
$$\mathbf{p}_i^{(0)} = \mathbf{p}_i$$

For increasing j:

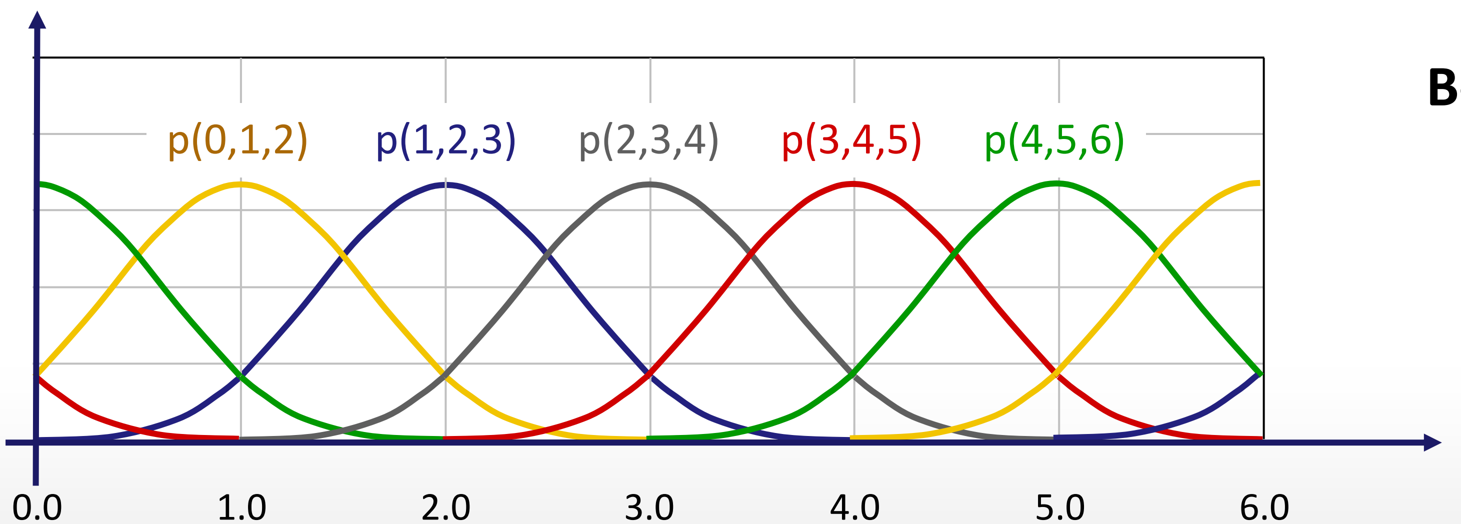
$$\mathbf{p}_i^{j}(t) = \alpha_i^{(j)}\mathbf{p}_i^{j-1}(t) + \left(1-\alpha_i^{(j)}\right)\mathbf{p}_{i-1}^{j-1}(t), \quad \alpha_i^{(j)} = \frac{\left(t - t_{i-1}\right)}{t_{i+d-j-1} - t_{i-1}}$$

Output $\mathbf{p}_i^{d-1}(t)$

Structure

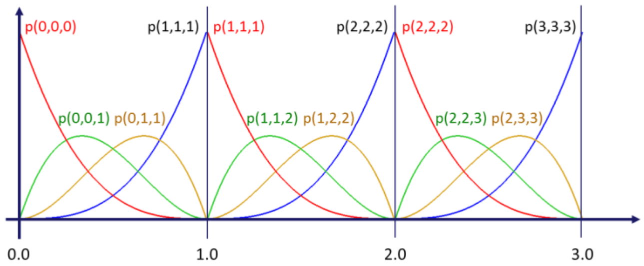


Bezier Spline

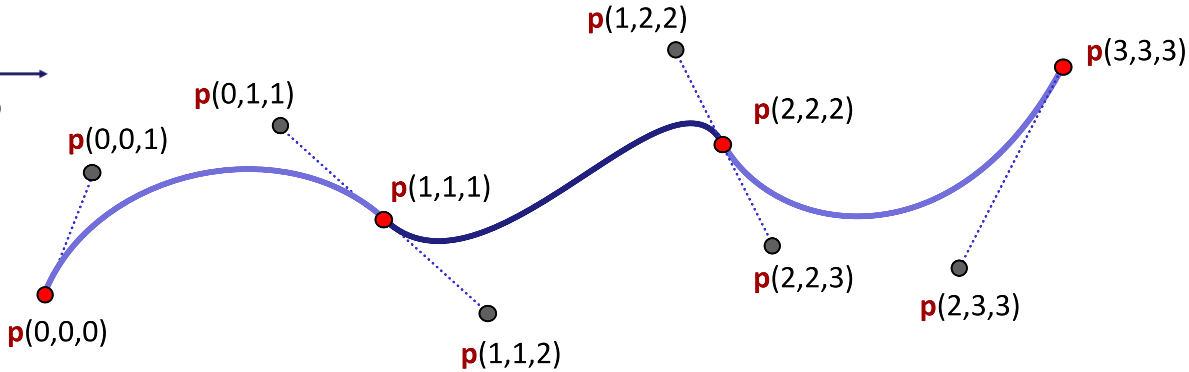


B-Spline

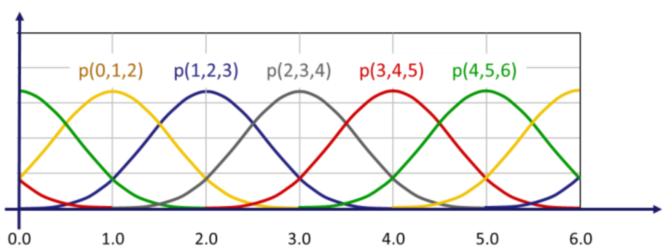
Structure

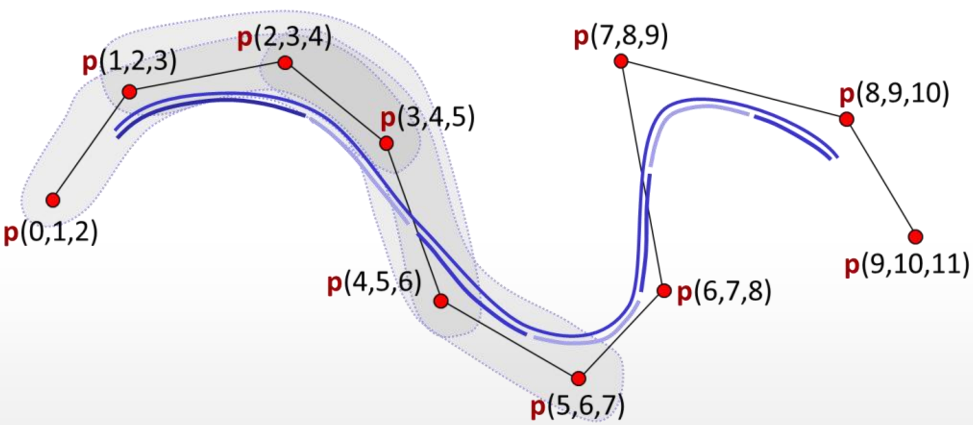


Bezier Spline



B-Spline

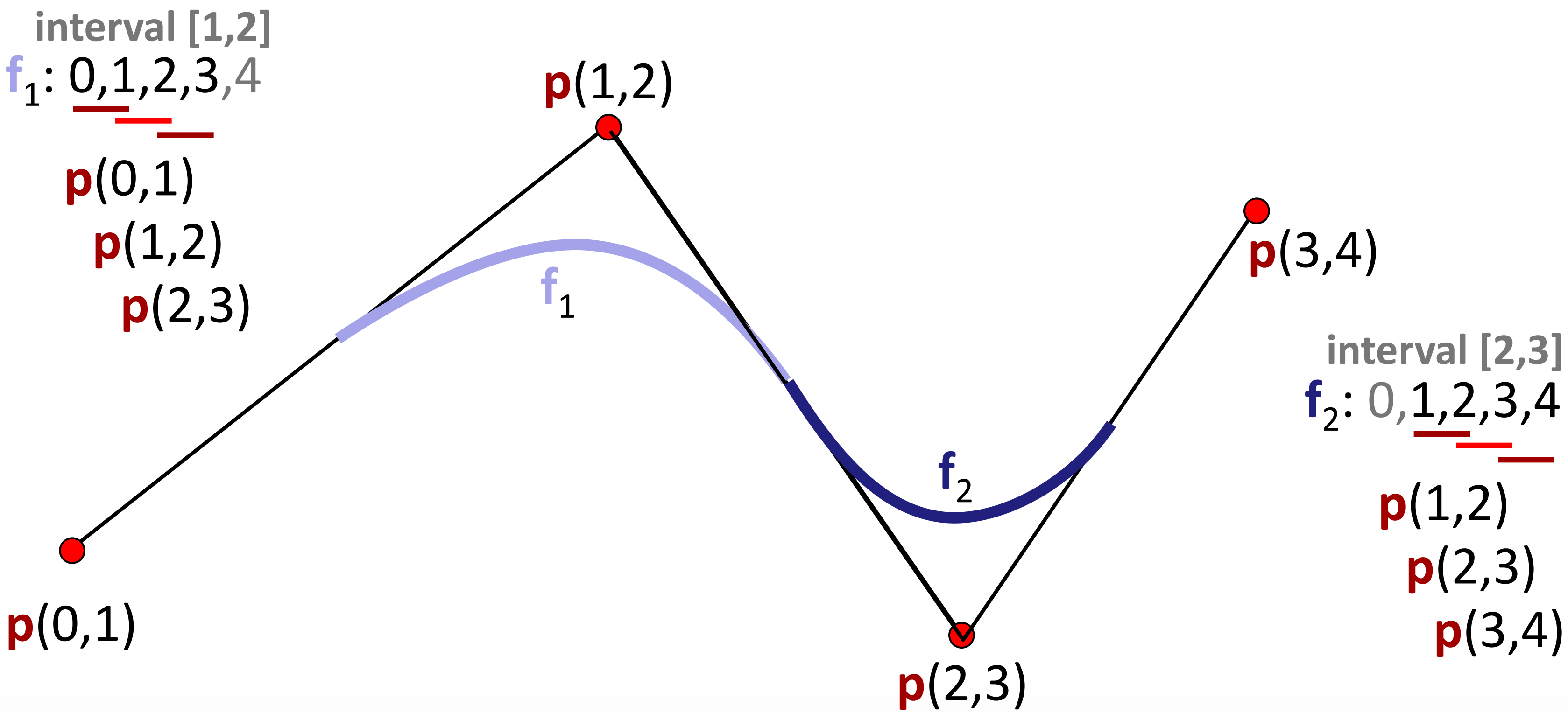




Smoothness

Multiplied Control Points:

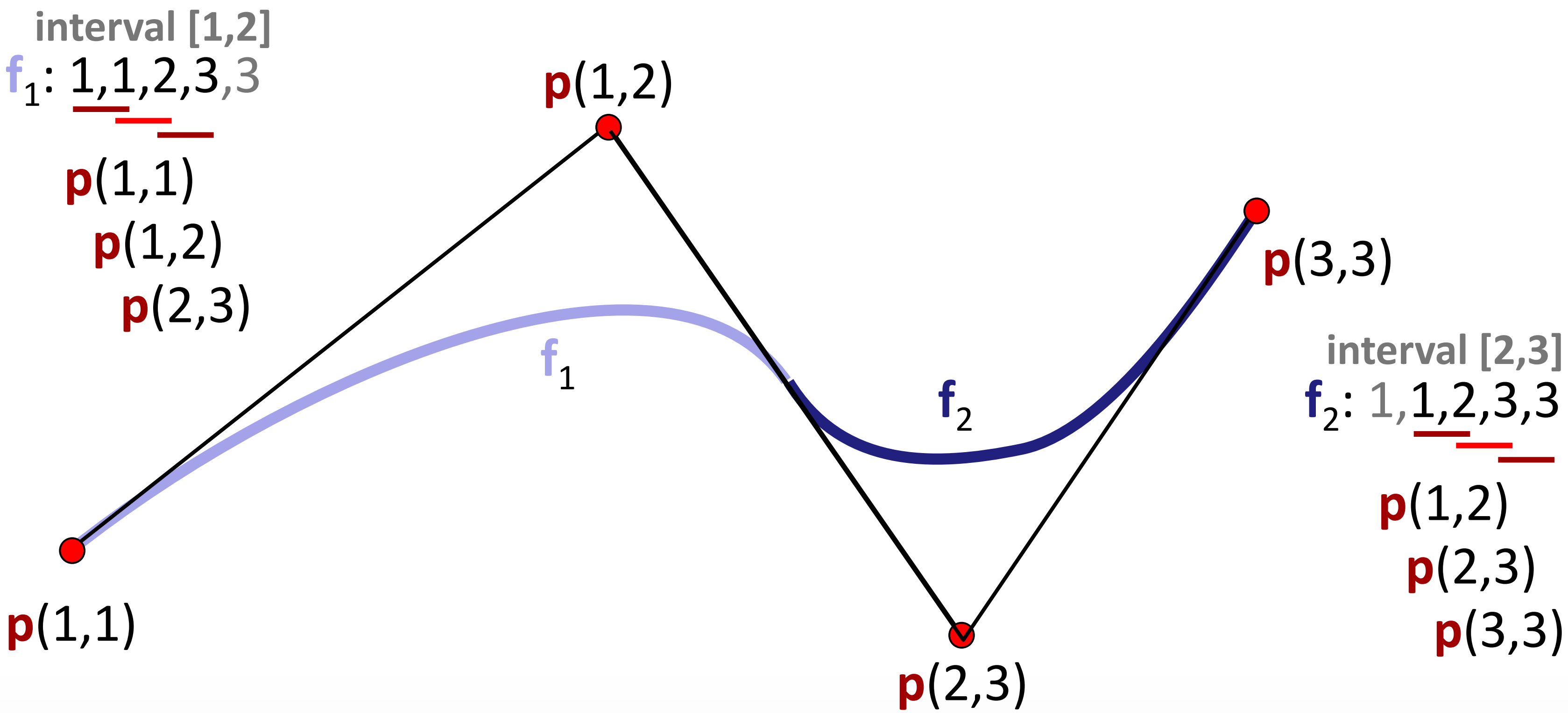
- Quadratic B-Spline Curve (two segments)



Smoothness

Multiplied Control Points:

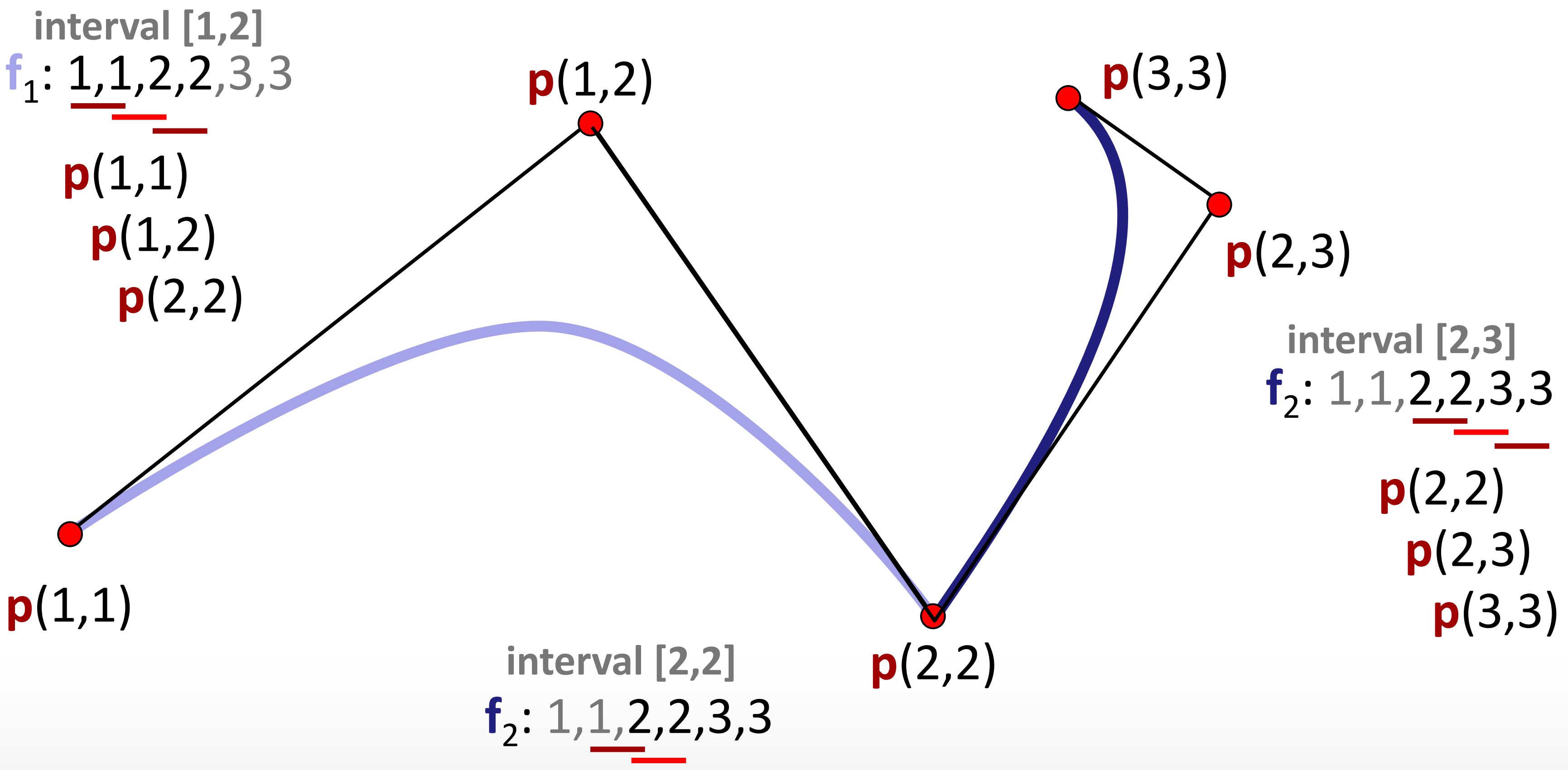
- Double end points $\rightarrow$ interpolated (c.f. de Casteljau)



Smoothness

Multiplied Control Points:

- Double inner node: reduces continuity by one.



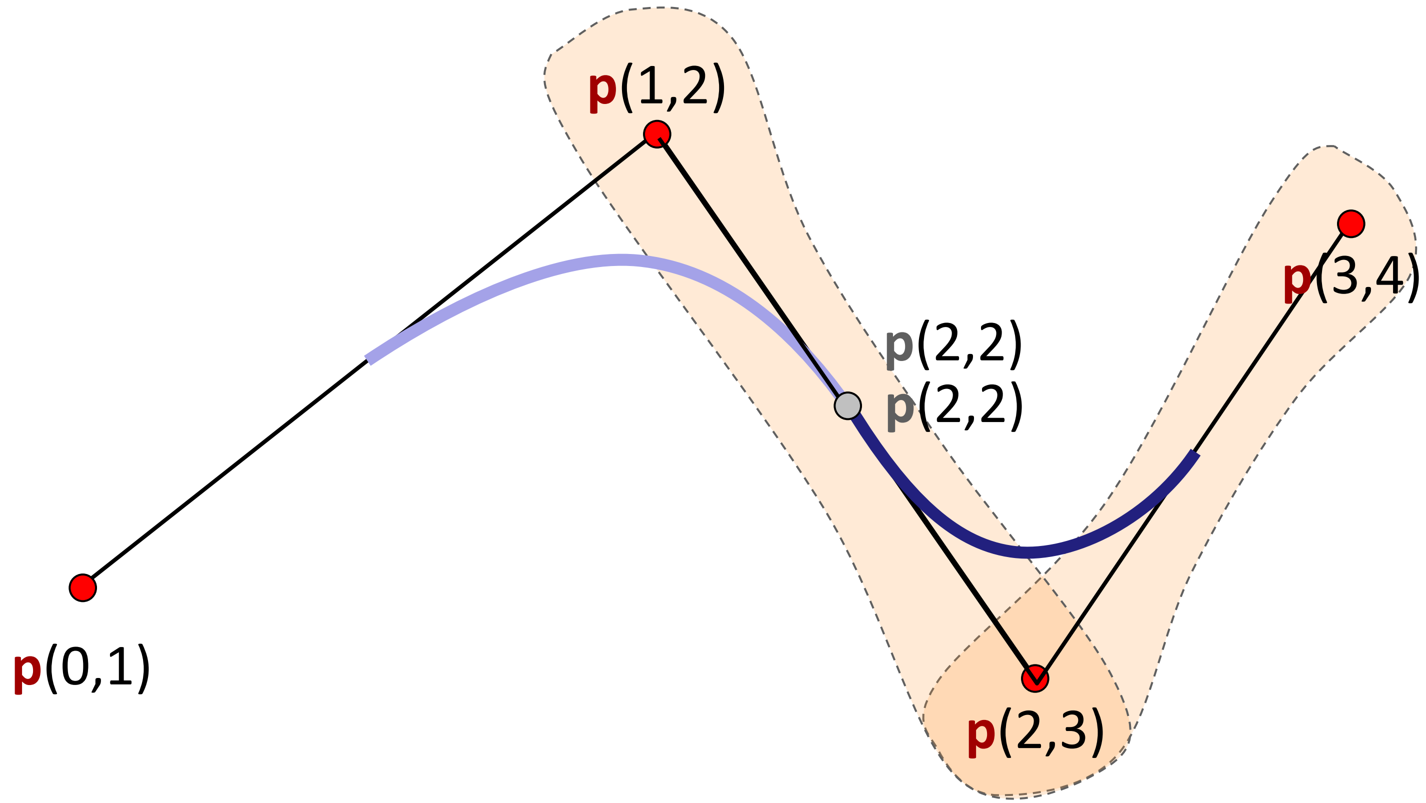
Smoothness

Multiplied Control Points: Blossoming

- Assume a control point contains one knot value *v* with multiplicity *k*.
- Through repeated linear interpolation, we can transform the blossoms for the left and right spline segment to:

  $\mathbf{p}(\underbrace{v,..,v}_{k},v_1,..,v_{d-k})$ and $\mathbf{p}(\underbrace{v,..,v}_{k},v_1,..,v_{d-k})$.
- This means, we have at least C^{d-k} continuity.

Example



$\mathbf{f}_1(0,1) = \mathbf{p}(0,1)$

$\mathbf{f}_1(1,2) = \mathbf{p}(1,2)$

$\mathbf{f}_1(2,3) = \mathbf{p}(2,3)$

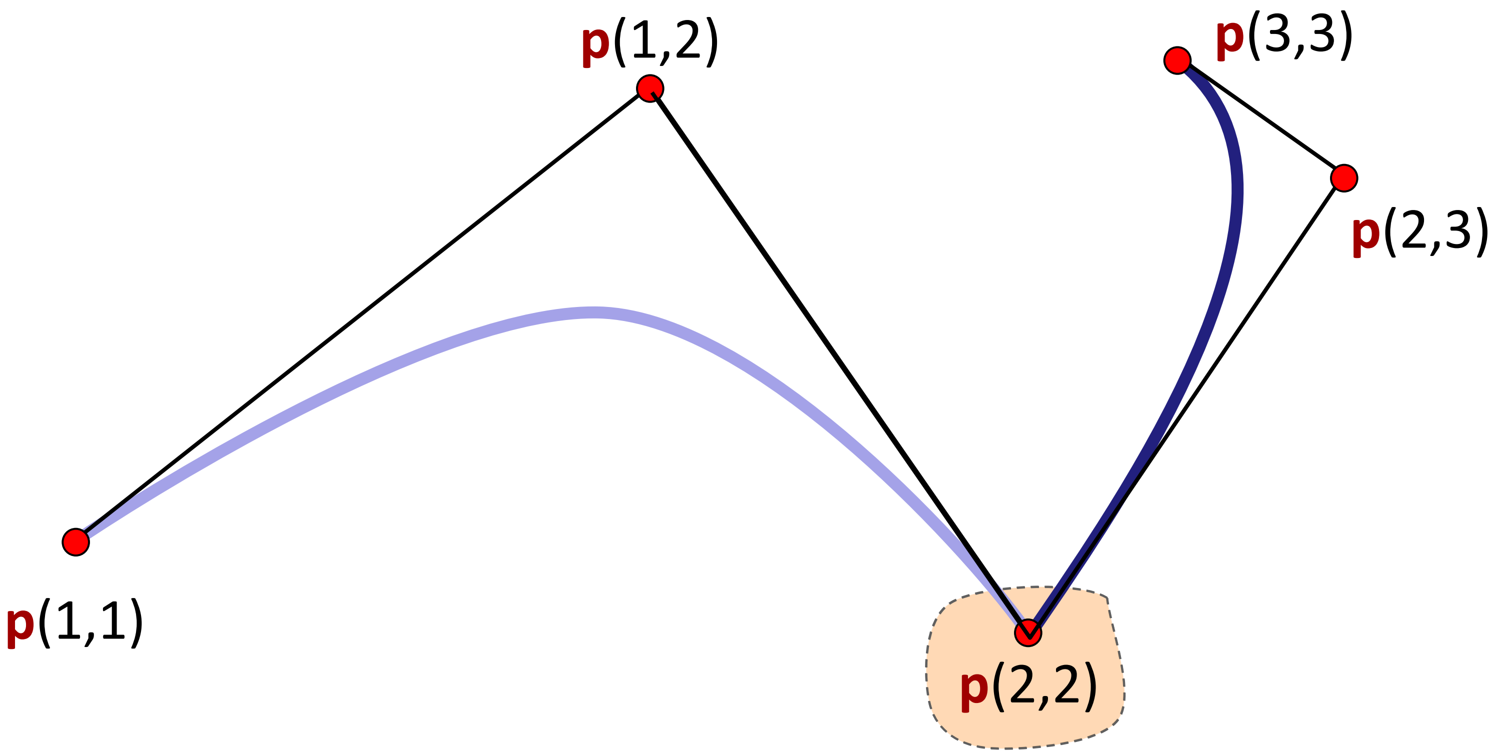
$\mathbf{f}_2(1,2) = \mathbf{p}(1,2)$

$\mathbf{f}_2(2,3) = \mathbf{p}(2,3)$

$\mathbf{f}_2(3,4) = \mathbf{p}(3,4)$

$\Rightarrow \quad \mathbf{f}_1(t,2) = \mathbf{f}_2(t,2)$

Smoothness



$$
\begin{array}{lll}
f_1(1,1) = \mathbf{p}(1,1) & f_2(2,2) = \mathbf{p}(2,2) & \\
f_1(1,2) = \mathbf{p}(1,2) & f_2(2,3) = \mathbf{p}(2,3) & \Rightarrow \quad f_1(2,2) = f_2(2,2) \\
f_1(2,2) = \mathbf{p}(2,2) & f_2(3,3) = \mathbf{p}(3,3) &
\end{array}
$$

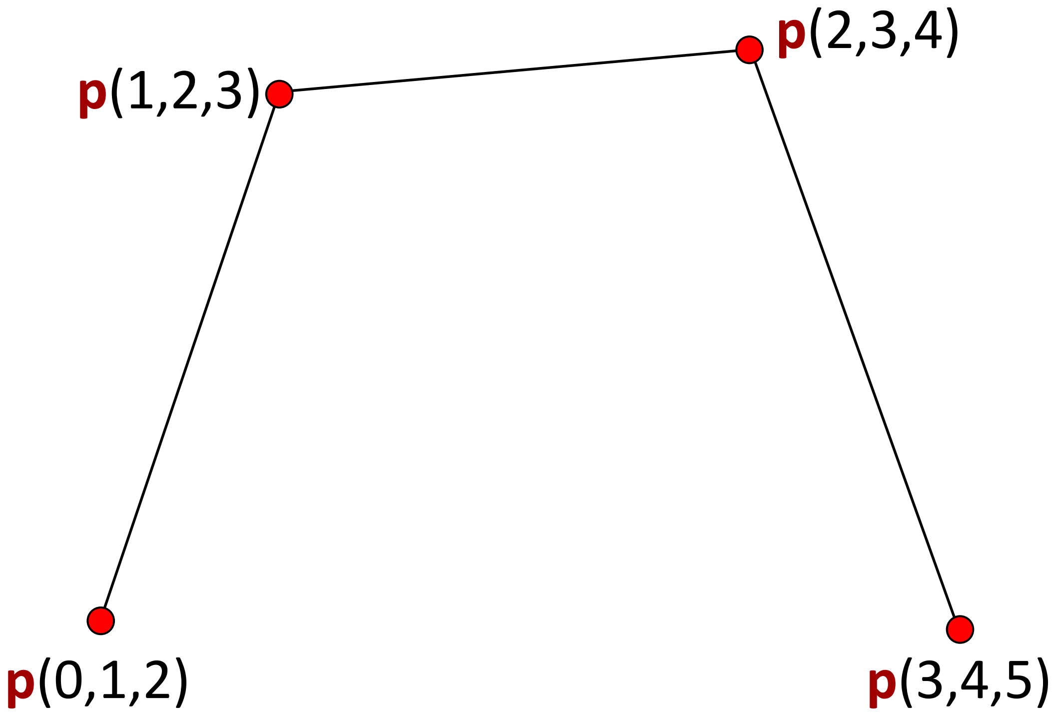
Connection to Bezier Splines

Special case:

- We can model Bezier splines through non-uniform B-Splines using a knot sequence $[\underbrace{a,\ldots,a}_{d}, \underbrace{b,\ldots,b}_{d}]$.
- For this knot sequence, the de Boor algorithm yields exactly the de Castlejau Algorithm.
- Therefore: the de Boor algorithm is a generalization of the de Casteljau algorithm.
- We can also insert Bezier segments somewhere into a B-spline.
- Interpolating end conditions are “half a Bezier segment”.

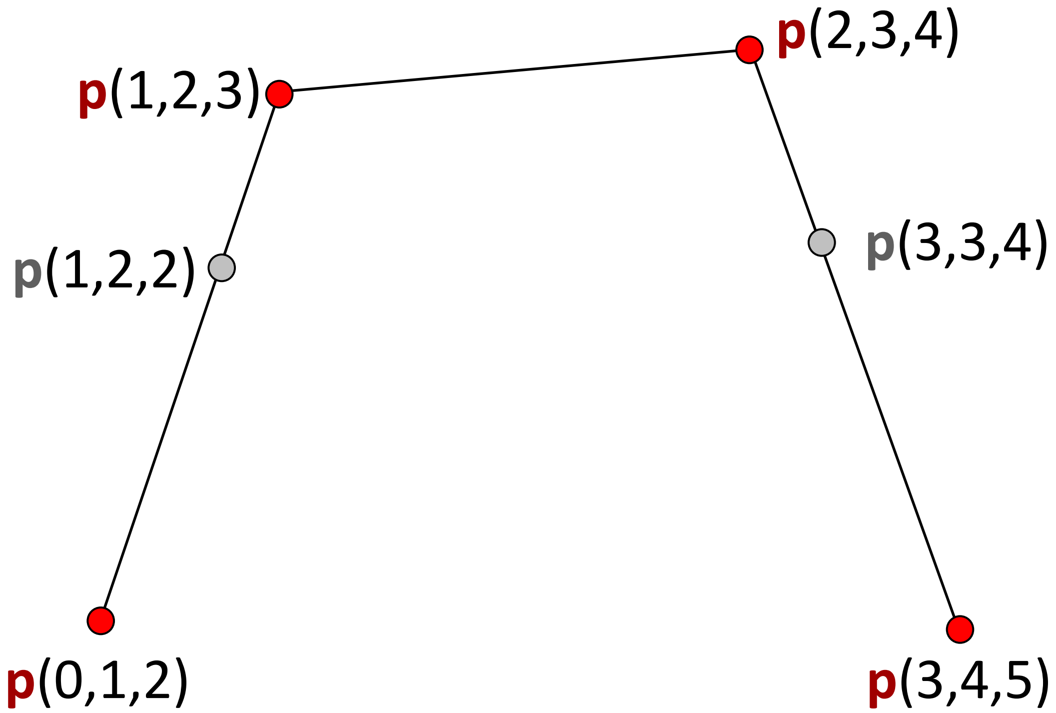
Conversion to Bezier Splines

Converting a B-spline segment to a Bezier segment:



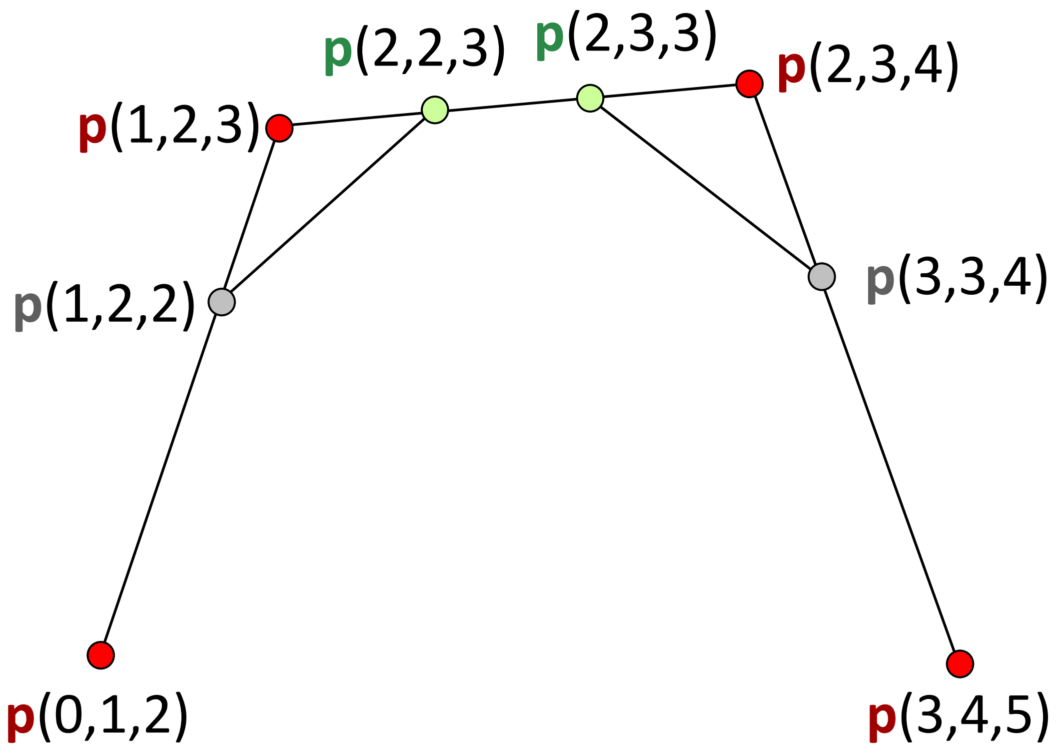
Conversion to Bezier Splines

Converting a B-spline segment to a Bezier segment:



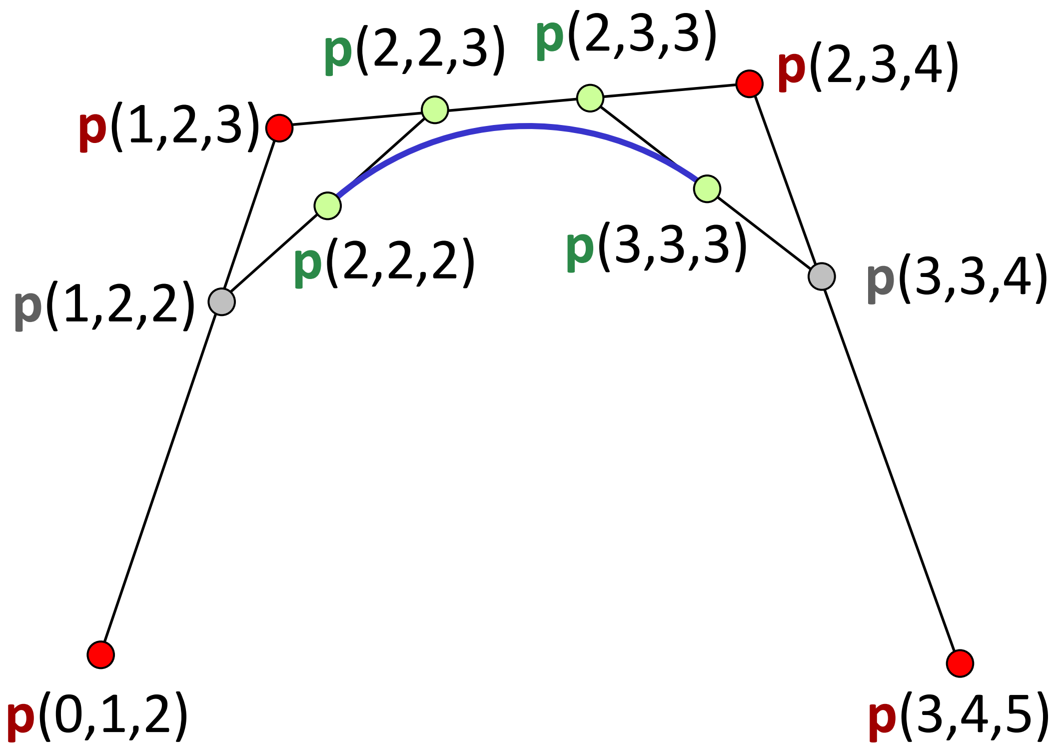
Conversion to Bezier Splines

Converting a B-spline segment to a Bezier segment:



Conversion to Bezier Splines

Converting a B-spline segment to a Bezier segment:



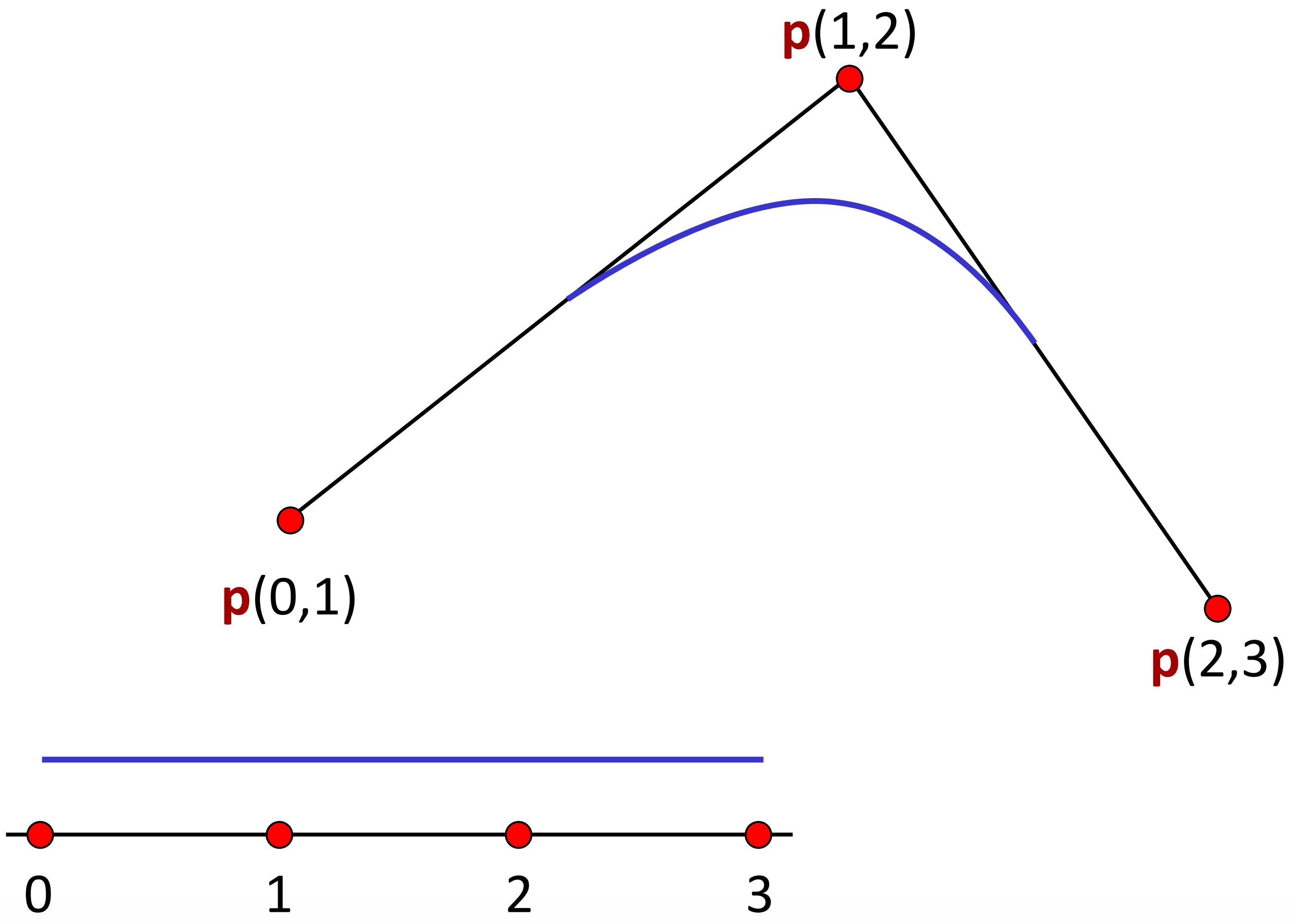
Knot Insertion

Problem:

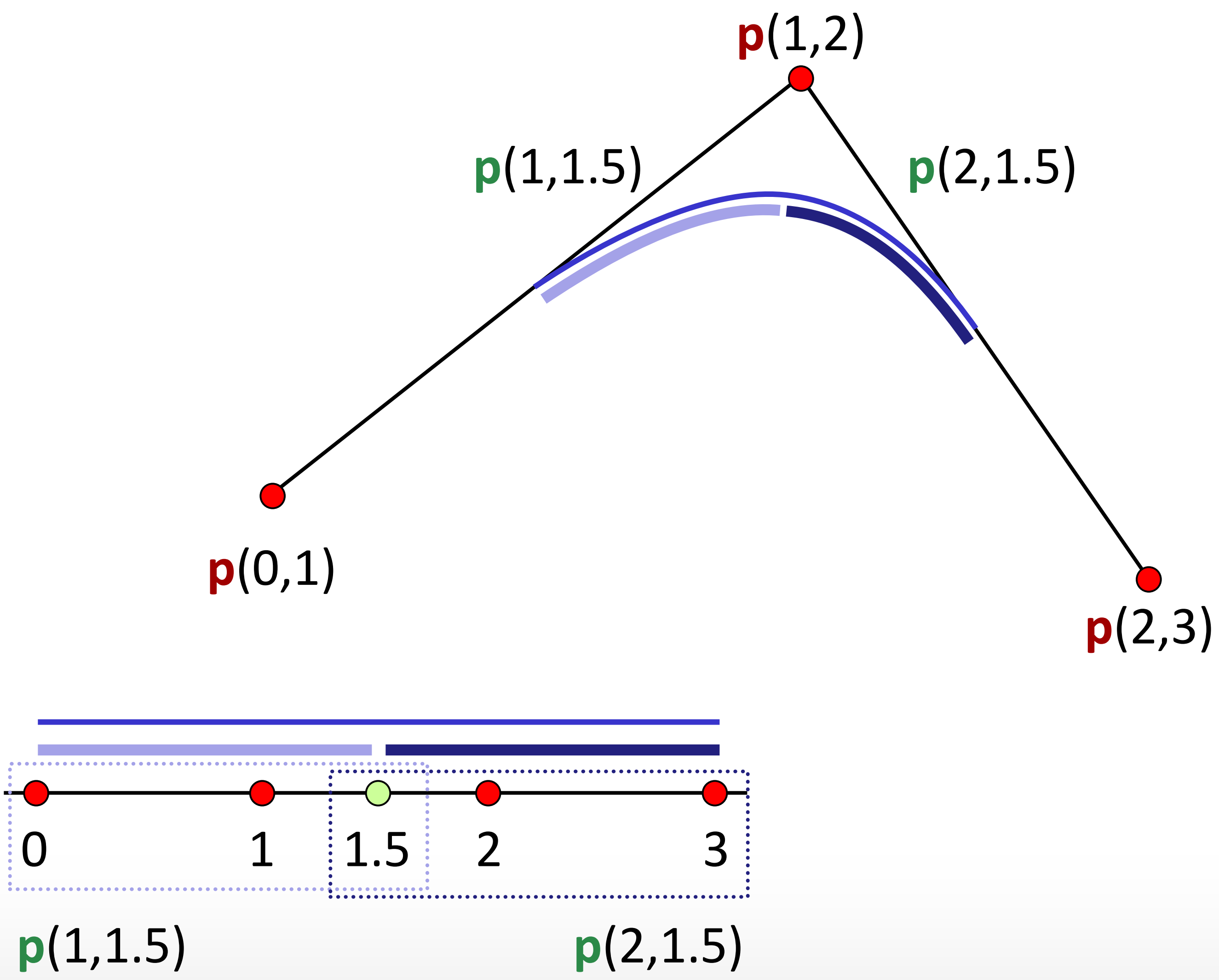
- Given a B-spline curve
- Insert a new knot value & a new control point in between two existing ones
- Without changing the curve

Very simple solution using blossoms

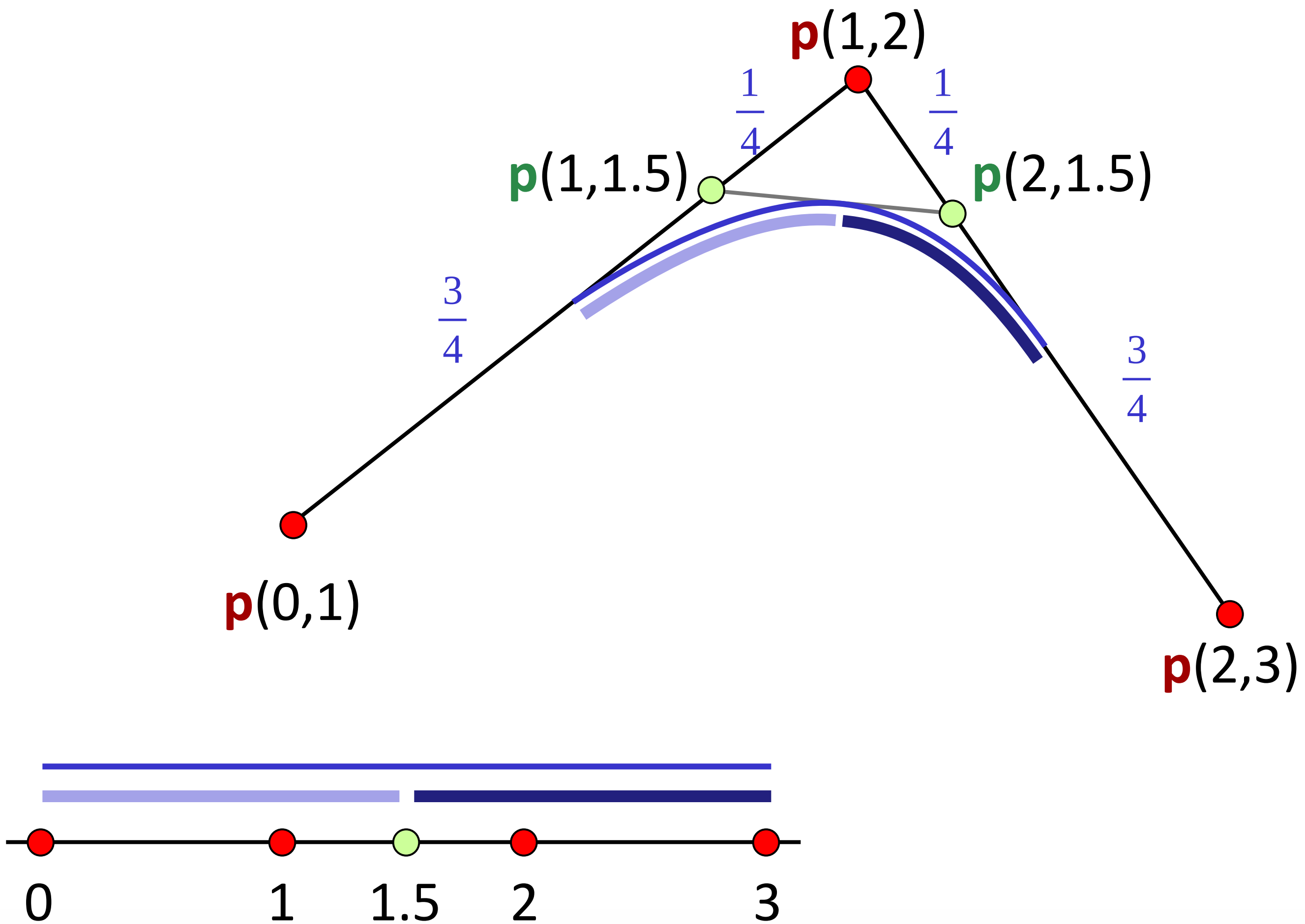
Quadratic Example



Quadratic Example



Quadratic Example



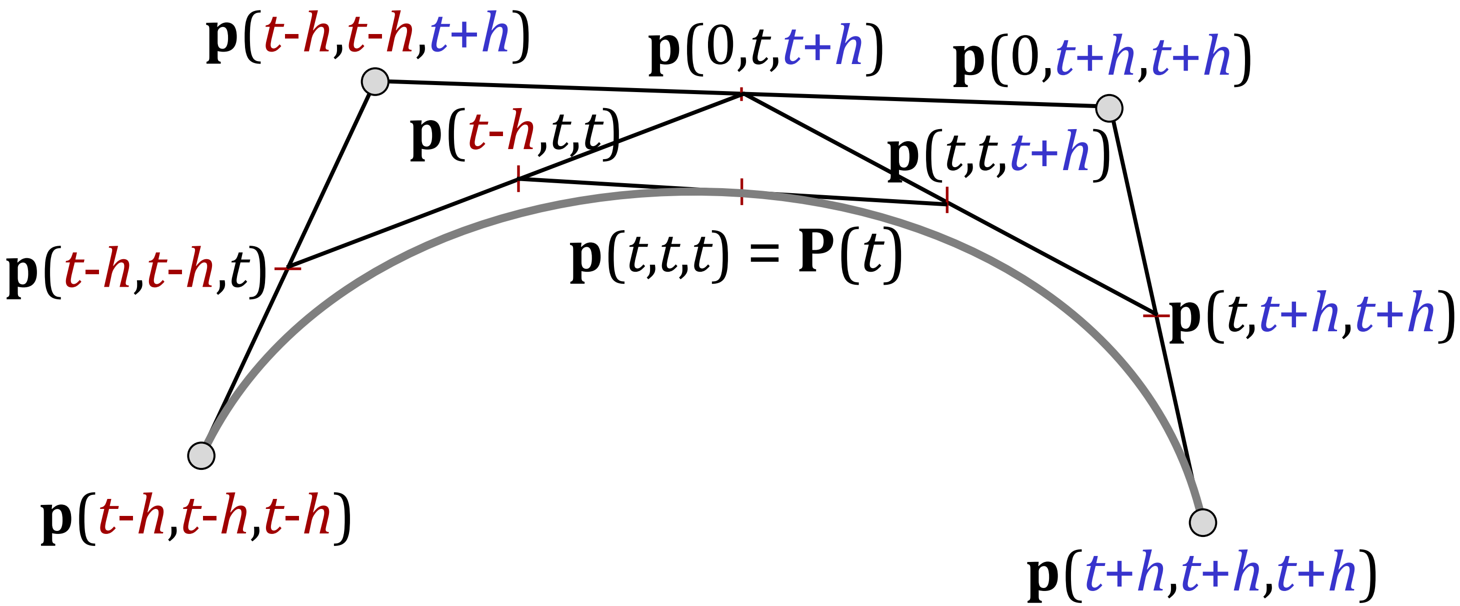
General Algorithm

The general case:

- Interval with knot to be added: $(t_i, \ldots, t_{i+2d-1})$
- Execute one step of the de Boor algorithm
- This creates new control points
  - Keep the outer points $p(t_i, \ldots, t_{i+d-1})$, $p(t_{i+1}, \ldots, t_{i+2d-1})$
  - Replace the inner points $p(t_{i+1}, \ldots, t_{i+d})$, $p(t_i, \ldots, t_{i+d-2})$ with the newly created points from the de Boor step
- This will...
  - Insert one knot value
  - And insert one additional control point
  - Change the existing control points, except the outermost

Visualization of Blossom Derivatives

De Casteljau



$$\begin{aligned}\frac{d\mathbf{P}(t)}{dt} &= \lim_{h\to 0}\frac{\mathbf{P}(t+h)-\mathbf{P}(t-h)}{2h}\\ &= \lim_{h\to 0}\frac{\mathbf{p}(t+h,t+h,t+h)-\mathbf{p}(t-h,t-h,t-h)}{2h}\\ &= \lim_{h\to 0}3\,\frac{\mathbf{p}(t,t,t+h)-\mathbf{p}(t,t,t-h)}{2h}\end{aligned}$$