Statistical Geometry Processing

Winter Semester 2011/2012

Differential Geometry
Multi-Dimensional Derivatives
Reminder: The *derivative* of a function is defined as

\[
\frac{d}{dt} f(t) := \lim_{h \to 0} \frac{f(t + h) - f(t)}{h}
\]

If limit exists: function is called *differentiable*.

Other notation:

\[
\frac{d}{dt} f(t) = f'(t) = \dot{f}(t)
\]

\[
\frac{d^k}{dt^k} f(t) = f^{(k)}(t)
\]

repeated differentiation (higher order derivatives)
Taylor Approximation

Smooth functions can be approximated locally:

- \( f(x) \approx f(x_0) \)
  \[ \quad + \frac{d}{dx} f(x_0) (x - x_0) \]
  \[ \quad + \frac{1}{2} \frac{d^2}{dx^2} f(x_0) (x - x_0)^2 + ... \]
  \[ \quad + \frac{1}{k!} \frac{d^k}{dx^k} f(x_0) (x - x_0)^k + O(x^{k+1}) \]

- Convergence: holomorphic functions
- Local approximation for smooth functions
Rule of Thumb

Derivatives and Polynomials

- Polynomial: \( f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \ldots \)
  - 0th-order derivative: \( f(0) = c_0 \)
  - 1st-order derivative: \( f'(0) = c_1 \)
  - 2nd-order derivative: \( f''(0) = 2c_2 \)
  - 3rd-order derivative: \( f'''(0) = 6c_3 \)
  - ...  

Rule of Thumb:

- Derivatives correspond to polynomial coefficients
- Estimate derivates \( \leftrightarrow \) polynomial fitting
Regularization

- Numerical differentiation needs regularization
  - Higher order is more problematic
- Finite differences (larger $h$)
- Averaging (polynomial fitting) over finite domain
Partial Derivative

Multivariate functions:

• Notation changes:

\[ \frac{\partial}{\partial x_k} f(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) := \]

\[ \lim_{h \to 0} \frac{f(x_1, \ldots, x_{k-1}, x_k + h, x_{k+1}, \ldots, x_n) - f(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n)}{h} \]

• Alternative notation:

\[ \frac{\partial}{\partial x_k} f(x) = \partial_k f(x) = f_{x_k}(x) \]
Special Cases

Derivatives for:

- Functions $f: \mathbb{R}^n \to \mathbb{R}$ (“heightfield”)
- Functions $f: \mathbb{R} \to \mathbb{R}^n$ (“curves”)
- Functions $f: \mathbb{R}^n \to \mathbb{R}^m$ (general case)
Special Cases

Derivatives for:

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• Functions \( f: \mathbb{R}^n \rightarrow \mathbb{R}^m \) (general case)
Gradient

Gradient:

• Given a function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) ("heightfield")

• The vector of all partial derivatives of \( f \) is called the \textit{gradient}:

\[
\nabla f(\mathbf{x}) = \begin{pmatrix}
\frac{\partial}{\partial x_1} f(\mathbf{x}) \\
\vdots \\
\frac{\partial}{\partial x_n} f(\mathbf{x})
\end{pmatrix}
\]
Gradient

Gradient:

- gradient: vector pointing in direction of steepest ascent.
- Local linear approximation (Taylor):
  \[ f(x) \approx f(x_0) + \nabla f(x_0) \cdot (x - x_0) \]
Higher order Derivatives:

- Can do all combinations: \( \frac{\partial}{\partial x_i_1} \frac{\partial}{\partial x_i_2} \cdots \frac{\partial}{\partial x_i_k} f \)

- Order does not matter for \( f \in C^k \)
Hessian Matrix

Higher order Derivatives:

• Important special case: Second order derivative

\[
\begin{pmatrix}
\frac{\partial^2}{\partial x_1^2} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} & \frac{\partial}{\partial x_1} \\
\frac{\partial}{\partial x_1} & \frac{\partial^2}{\partial x_2^2} & \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} & \frac{\partial}{\partial x_2} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_n} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial^2}{\partial x_n^2}
\end{pmatrix}
\]

\[f(x) = H_f(x)\]

• “Hessian” matrix (symmetric for \( f \in C^2 \))

• Orthogonal Eigenbasis, full Eigenspectrum
Taylor Approximation

Second order Taylor approximation:

- Fit a paraboloid to a general function

\[ f(x) \approx f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2} (x - x_0)^T \cdot H_f(x_0) \cdot (x - x_0) \]
Special Cases

Derivatives for:

- Functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ ("heightfield")
- Functions $f: \mathbb{R} \rightarrow \mathbb{R}^n$ ("curves")
- Functions $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ (general case)
Derivatives of Curves

Derivatives of vector valued functions:

• Given a function $f: \mathbb{R} \rightarrow \mathbb{R}^n$ (“curve”)

$$f(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

• We can compute derivatives for every output dimension:

$$\frac{d}{dt} f(t) := \begin{pmatrix} \frac{d}{dt} f_1(t) \\ \vdots \\ \frac{d}{dt} f_n(t) \end{pmatrix} =: f'(t) =: \dot{f}(t)$$
Geometric Meaning

Tangent Vector:

- \( f' \): tangent vector
- Motion of physical particle: \( \dot{f} = \text{velocity} \).
- Higher order derivatives: Again vector functions
- Second derivative \( \ddot{f} = \text{acceleration} \)
Special Cases

Derivatives for:

- Functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ ("heightfield")
- Functions $f: \mathbb{R} \rightarrow \mathbb{R}^n$ ("curves")
- Functions $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ (general case)
You can combine it...

General case:

- Given a function \( f : \mathbb{R}^n \to \mathbb{R}^m \) (“space warp”)

\[
\begin{pmatrix}
 f_1(x_1, \ldots, x_n) \\
 \vdots \\
 f_m(x_1, \ldots, x_n)
\end{pmatrix}
\]

- Maps points in space to other points in space

- First derivative: Derivatives of all output components of \( f \) w.r.t. all input directions.

- “Jacobian matrix”: denoted by \( \nabla f \) or \( J_f \)
Jacobian Matrix

Jacobian Matrix:

\[ \nabla f(x) = J_f(x) = \nabla f(x_1, \ldots, x_n) \]

\[ = \begin{pmatrix} \nabla f_1(x_1, \ldots, x_n)^T \\ \vdots \\ \nabla f_m(x_1, \ldots, x_n)^T \end{pmatrix} = \begin{pmatrix} \partial_{x_1} f_1(x) & \cdots & \partial_{x_n} f_1(x) \\ \vdots & & \vdots \\ \partial_{x_1} f_m(x) & \cdots & \partial_{x_n} f_m(x) \end{pmatrix} \]

Use in a first-order Taylor approximation:

\[ f(x) \approx f(x_0) + J_f(x_0)(x - x_0) \]

matrix / vector product
Coordinate Systems

Problem:

- What happens, if the coordinate system changes?
- Partial derivatives go into different directions then.
- Do we get the same result?
Total Derivative

First order Taylor approx.: 

- \( f(x_0) + \nabla f(x_0) \cdot (x - x_0) + R_{x_0}(x) \)
- Converges for \( C^1 \) functions \( f: \mathbb{R}^n \to \mathbb{R}^m \)

\[
\lim_{x \to x_0} \frac{R_{x_0}(x)}{\|x - x_0\|} = 0,
\]

(“totally differentiable”)
Partial Derivatives

Consequences:

- A linear function: fully determined by image of a basis
- Hence: Directions of partial derivatives do not matter – this is just a basis transform.
  - We can use any linear independent set of directions $T$
  - Transform to standard basis by multiplying with $T^{-1}$
- Similar argument for higher order derivatives
The directional derivative is defined as:

- Given $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $v \in \mathbb{R}^n$, $||v|| = 1$.

- Directional derivative:
  \[
  \nabla_v f(x) = \left( \frac{\partial f}{\partial v}(x) := \frac{d}{dt} f(x + tv) \right)
  \]

- Compute from Jacobian matrix
  \[
  \nabla_v f(x) = \nabla f(x) \frac{v}{||v||}
  \]

  (requires total differentiability)
Multi-Dimensional Optimization
Optimization Problem:

- Given a $C^1$ function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ (general heightfield)
- We are looking for a local extremum (minimum / maximum) of this function

Theorem:

- $x$ is a local extremum $\Rightarrow \nabla f(x) = 0$

Sketch of a proof: If $\nabla f(x) \neq 0$, we can walk a small step in gradient direction to improve the score further (in case of a maximum, minimum similar).
Critical Points

Critical points:

- $\nabla f(x) = 0$ does not guarantee an extremum (saddle points)
- Points with $\nabla f(x) = 0$ are called critical points.
- Final decision via Hessian matrix:
  - All eigenvalues > 0: local minimum
  - All eigenvalues < 0: local maximum
  - Mixed eigenvalues: saddle point
  - Some zero eigenvalues: critical line
Quadratic Optimization

Quadratic Case:

- $f: \mathbb{R}^n \rightarrow \mathbb{R}$
- Objective function: $f(x) = x^T Ax + b^T x + c$
  - symmetric $n \times n$ matrix $A$
  - $n$-dim. vector $b$
  - constant $c$
- Gradient: $\nabla f(x) = 2Ax + b$
- Critical points: solution to $2Ax = -b$
- Solution: Solve system of linear equations
**Example**

**Gradient computation example:**

\[
[x, y] \begin{pmatrix} a \\ b \end{pmatrix} = ax + by \rightarrow \nabla = \begin{pmatrix} a \\ b \end{pmatrix}
\]

\[
[x, y] \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2
\]

\[
\partial_x \rightarrow 2ax + 2by \\
\partial_y \rightarrow 2bx + 2cy
\]

\[
\nabla \rightarrow 2A \begin{pmatrix} x \\ y \end{pmatrix}
\]
Global Extrema of Quadratic Funcs.

Three cases:

- Eigenvalues of $\mathbf{A} \geq 0$: critical points are global minima
- Eigenvalues of $\mathbf{A} \leq 0$: critical points are global maxima
- Mixed eigenvalues: no global minimum/maximum exists (minimum and maximum at infinity)

Structure:

- Critical points form an affine subspace of $\mathbb{R}^n$.
- I.e.: Point, line, plane...
Non-Linear Optimization Algorithms
Non-Quadratic Optimization

Optimization Problems:

• Find (local/global) minimum of $E: \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R}$.

• $E$ for “energy” (motivated from physics)

• What to do if $E$ is non-quadratic?
Gradient Descent:

- Gradient $\nabla E$ points into direction of steepest ascent.
- Walking a small step in direction $-\nabla E$ will decrease the energy.
- When $\nabla E = 0$, a critical point is found.

Properties:

- For sufficiently small steps, this algorithm is guaranteed to converge
- Generally slow convergence
- Does not work in practice for ill-conditioned problems
Newton Optimization

• Basic idea: Local quadratic approximation of $E$:

$$E(x) \approx E(x_0) + \nabla E(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0)^T \cdot H_E(x_0) \cdot (x - x_0)$$

• Solve for vertex (critical point) of the fitted parabola

• Iterate until a minimum is found ($\nabla E = 0$)

Properties:

• Typically much faster convergence, more stable

• No convergence guarantee
Newton Optimization - Divergence

Regularization:

- Hessian matrix: for negative eigenvalues, steps might point uphill
- (Near-) zero eigenvalues make problem ill-conditioned.
- Simple solution: Add $\lambda I$ to the Hessian for a small $\lambda$.
- Sum of two quadrics: $\lambda I$ keeps solution at $x_0$.
- This is an example of regularization
Handling Indefinite Situations

Initial state:

\[ \text{minimum } \times \]

\[ H_E \]

First Iteration:

\[ \text{minimum } \times \text{new solution } \]

\[ H_E \times \]

\[ H_E + \lambda I \]

New state:

\[ \text{minimum } \times \]

\[ H_E \times \]

\[ H_E + \lambda I \]

Second Iteration:

\[ \text{minimum } \times \text{new solution } \]

\[ H_E \times \]

\[ H_E + \lambda I \]

...
Further Algorithms

Gradient descent line search:

- Optimize step size for gradient descent
  - Fit 1D parabola to $E$ in gradient direction
  - Perform 1D Newton search
  - If $E$ does not decrease at the new position:
    - Try to half step width (say up to 10-20 times).
    - If this still does not decrease $E$, stop and output local minimum.
Further Algorithms

Line search for Newton-optimization:

- Following the quadratic fit might overshoot
- Line search:
  - Test value of $E$ at new position
  - Half step width until error decreases (say 10-20 iterations)
  - Switch to gradient descent, if this does not work
Convex Problems

General Classification:

- Non-linear optimization problems can be hard to solve.
- What is definitely “easy”? 

Convex Problems:

- *Convex functions* on a *convex domain* can be optimized “easily” using a generic algorithm.
- Other problems *might* be hard to solve.
Convex Function:

- A $C^2$ function $E$ is convex, if $H_E > 0$ (all eigenvalues of the Hessian are strictly positive everywhere).
- A set $\Omega$ is convex if every line connecting two points from $\Omega$ is also contained in $\Omega$.
- A convex function has at most one local minimum.

Problem Properties:

- Assume a global minimum exists.
- Will be the only local minimum.
- Can be reached on a straight line from any point in $\Omega$. 
Convex Problems

Generic Optimization Algorithm (Sketch):

• Gradient descent

• Start at any point $p \in \Omega$

• Perform gradient descent in “small enough” steps

• In case of hitting the domain boundary, project on boundary surface (follow the wall)

• When the gradient becomes zero, the minimum is found

There are more efficient algorithms...
Multi-Dimensional Integrals
Integral

Integral of a function

• Function $f: \mathbb{R} \rightarrow \mathbb{R}$

• Integral $\int_{a}^{b} f(t)dt$ measures signed area under curve:
Integral

Numerical Approximation

• Sum up a series of approximate shapes

(Riemannian) Definition: limit for baseline $\rightarrow$ zero
Multi-Dimensional Integral

Integration in higher dimensions

- Functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$
- Tessellate domain and sum up volume of cuboids
Integral Transformations

Integration by substitution:

\[
\int_{a}^{b} f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))g'(t)dt
\]

Need to compensate for speed of movement that shrinks the measured area.
Multi-Dimensional Substitution

Transformation of Integrals:

\[ \int_{\Omega} f(x) dx = \int_{g^{-1}(\Omega)} f(g(y)) |\text{det}(\nabla g(y))| \, dy \]

- \( g \in C^1 \), invertible
- Jacobian approximates local behavior of \( g() \)
- Determinant: local area/volume change
- In particular: \(|\text{det}(\nabla g(y))| = 1\) means \( g() \) is area/volume conserving.
Topology
- a very short primer -
A Few Concepts from Topology

Homeomorphism:

- $f: X \rightarrow Y$
- $f$ is bijective
- $f$ is continuous
- $f^{-1}$ exists and is continuous
- Basically, a continuous deformation

Topological equivalence

- Objects are topologically equivalence if there exists a homeomorphism that maps between them
- “Can be deformed into each other”
Surfaces

Boundaries of volumes in 3D

• Topological Equivalence classes
  ▪ Sphere
  ▪ Torus
  ▪ n-fold Torus

• Genus = number of tunnels

\[ g = 0 \quad g = 1 \quad g = 2 \]
Definition: Manifold

- A $d$-manifold $M$:
  At every $x \in M$ there exists an $\epsilon$-environment homeomorphic to a $d$-dimensional disc
- With boundary: disc or half-disc
Further concepts

Connected Set
- There exists a continuous curve within the set between all pairs of points

Simply Connected
- Every closed loop can be continuously shrunken until it disappears
Differential Geometry
of Curves & Surfaces
Part I: Curves
Parametric Curves:

- A differentiable function
  \[ f: (a, b) \rightarrow \mathbb{R}^n \]
  describes a *parametric curve*
  \[ C = f((a, b)), \quad C \subseteq \mathbb{R}^n. \]

- The parametrization is called *regular* if \( f'(t) \neq 0 \) for all \( t \).

- If \( \|f'(t)\| = 1 \) for all \( t \), \( f \) is called a *unit-speed parametrization* of the curve \( C \).
Length of a Curve

The length of a curve:

- The length of a regular curve \( C \) is defined as:
  \[
  \text{length}(C) = \int_a^b \| f'(t) \| \, dt
  \]

- Independent of the parametrization (integral transformation theorem).

- Alternative: \( \text{length}(C) = |b - a| \) for a unit-speed parametrization
Reparametrization

Enforcing unit-speed parametrization:

- Assume: $\|f'(t)\| \neq 0$ for all $t$.
- We have:
  $$\text{length}(C) = \int_{a}^{b} \|f'(t)\| dt$$  (invertible, because $f'(t) > 0$)
- Concatenating $f \circ \text{length}^{-1}(C)$ yields a unit-speed parametrization of the curve
Tangents

Unit Tangents:

• The unit tangent vector at $x \in (a, b)$ is given by:

$$
tangent(t) = \frac{f'(t)}{\|f'(t)\|}
$$

• For curves $C \subseteq \mathbb{R}^2$, the unit normal vector of the curve is defined as:

$$
\text{normal}(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{f''(t)}{\|f'(t)\|}
$$
Curvature

Curvature:

• First derivatives show curve direction / speed of movement.
• Curvature is encoded in 2nd order information.
• Why not just use $f''$?
• Problem: Depends on parametrization
  ▪ Different velocity yields different results
  ▪ Need to distinguish between acceleration in tangential and non-tangential directions.
Curvature & 2nd Derivatives

Definition of curvature

- We want only the non-tangential component of $f''$.
- Accelerating/slowing down does not matter for curvature of the traced out curve $C$.
- Need to normalize speed.
Curvature

Curvature of a Curve $C \in \mathbb{R}^2$:

$$\kappa_2(t) = \frac{\left\langle f''(t), \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} f'(t) \right\rangle}{\|f'(t)\|^3}$$

- Normalization factor:
  - Divide by $\|f'\|$ to obtain unit tangent vector
  - Divide again twice to normalize $f''$
    - Taylor expansion / chain rule:
      $$f(\lambda t) = f(t_0) + \lambda f'(t_0)(t - t_0) + \frac{1}{2} \lambda^2 f''(t_0)(t - t_0)^2 + O(t^3)$$
    - Second derivative scales quadratically with speed
Unit-speed parametrization:

• Assume a unit-speed parametrization, i.e. $\|f'\| = 1$.
• Then, $\kappa_2$ simplifies to:

$$\kappa_2(t) = \|f''(t)\|$$
Radius of Curvature

Easy to see:

• Curvature of a circle is constant, $\kappa^2 \equiv \pm 1/r$ ($r = \text{radius}$). (see problem sets)

• Accordingly: Define radius of curvature as $1/\kappa^2$.

• Osculating circle:
  
  ▪ Radius: $1/\kappa^2$
  ▪ Center: $f(t) + \frac{1}{\kappa^2} \text{normal}(t)$
Theorems

Definition:

• Rigid motion: \( x \rightarrow Ax+b \) with orthogonal \( A \)
  ▪ Orientation preserving (no mirroring) if \( \det(A) = +1 \)
  ▪ Mirroring leads to \( \det(A) = -1 \)

Theorems for plane curves:

• Curvature is invariant under rigid motion
  ▪ Absolute value is invariant
  ▪ Signed value is invariant for orientation preserving rigid motion

• Two unit speed parameterized curves with identical signed curvature function differ only in a orientation preserving rigid motion.
General case: Curvature of a Curve \( C \subseteq \mathbb{R}^n \)

- W.l.o.g.: Assume we are given a unit-speed parametrization \( f \) of \( C \)
- The *curvature* of \( C \) at parameter value \( t \) is defined as:
  \[
  \kappa(t) = \|f''(t)\|
  \]

- For a general, regular curve \( C \subseteq \mathbb{R}^3 \) (any regular parametrization):
  \[
  \kappa(t) = \frac{\|f'(t) \times f''(t)\|}{\|f'(t)\|^3}
  \]

- General curvature is unsigned
Torsion

Characteristics of Space Curves in $\mathbb{R}^3$:

- Curvature not sufficient
- Curve may “bend” in space
- Curvature is a 2nd order property
- 2nd order curves are always flat
  - Quadratic curves are specified by 3 points in space, which always lie in a plane
  - Cannot capture out-of-plane bends
- Missing property: Torsion
Torsion

Definition:

- Let $f$ be a regular parametrization of a curve $C \subseteq \mathbb{R}^3$ with non-zero curvature.
- The torsion of $f$ at $t$ is defined as

$$\tau(t) = \frac{f'(t) \times f''(t) \cdot f'''(t)}{\|f'(t) \times f''(t)\|^2} = \frac{\det(f'(t), f''(t), f'''(t))}{\|f'(t) \times f''(t)\|^2}$$
\[ \tau(t) = \frac{\det(f''(t), f'''(t), f''''(t))}{\|f'(t) \times f''(t)\|^2} \]
Theorem

Fundamental Theorem of Space Curves

- Two unit speed parameterized curves $C \subset \mathbb{R}^3$ with identical, positive curvature and identical torsion are identical up to a rigid motion.
Part II: Surfaces
Parametric Surface Patches:

A smoothly differentiable function

\[ f: \mathbb{R}^2 \supseteq \Omega \rightarrow \mathbb{R}^n \]

describes a parametric surface patch

\[ P = f(\Omega), \ P \subseteq \mathbb{R}^n. \]
Parametric Patches

Function $f(x) = f(u, v) \rightarrow \mathbb{R}^3$

- Tangents: $\frac{d}{dt} f(x_0 + tr) = \nabla_r f(x_0)$

- Canonical tangents: $\partial_u f(u,v), \partial_v f(u,v)$

- Normal: $n(x_0) = \frac{\partial_u f(u,v) \times \partial_v f(u,v)}{||\partial_u f(u,v) \times \partial_v f(u,v)||}$
Illustration

\[ \Omega \subseteq \mathbb{R}^2 \]

\[ P \subseteq \mathbb{R}^3 \]

normal \((u, v)\)

\[ \partial_v f(u, v) \]

\[ \partial_u f(u, v) \]
Surface Area

Surface Area:

- Patch $P : f : \Omega \rightarrow \mathbb{R}^3$
- Computation is simple
- Integrate over constant function $f \equiv 1$ over surface
- Then apply integral transformation theorem:

$$\text{area}(P) = \int_{\Omega} \left\| \partial_u f(x) \times \partial_u f(x) \right\| dx$$
Fundamental Forms:

- Describe the local parametrized surface
- Measure...
  - ...distortion of length (first fundamental form)
  - ...surface curvature (second fundamental form)
- Parametrization independent surface curvature measures will be derived from this
First Fundamental Form

- Also known as *metric tensor*.
- Given a regular parametric patch \( f: \mathbb{R}^2 \supseteq \Omega \to \mathbb{R}^3 \).
- \( f \) will distort angles and distances.
- We will look at a local first order Taylor approximation to measure the effect:
  \[
  f(x) \approx f(x_0) + \nabla f(x_0)(x - x_0)
  \]
- Length changes become visible in the scalar product...
First Fundamental Form

- First order Taylor approximation:
  \[ f(x) \approx f(x_0) + \nabla f(x_0) (x - x_0) \]
- Scalar product of vectors \( a, b \in \mathbb{R}^2 \):
  \[
  \langle f(x_0 + a) - f(x_0), f(x_0 + b) - f(x_0) \rangle \approx \langle \nabla f(x_0) a, \nabla f(x_0) b \rangle \\
  = a^T \left( \nabla f(x_0)^T \nabla f(x_0) \right) b
  \]
  first fundamental form
First Fundamental Form

The first fundamental form can be written as a $2 \times 2$ matrix:

$$(\nabla f^T \nabla f) = \begin{pmatrix} \frac{\partial u f}{\partial u} & \frac{\partial u f}{\partial v} \\ \frac{\partial v f}{\partial u} & \frac{\partial v f}{\partial v} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$I(x,y) := x^T (\nabla f^T \nabla f) y$

- The matrix is symmetric and positive definite (regular parametrization, semi-definite otherwise)
- Defines a **generalized scalar product** that measures lengths and angles *on the surface*. 

$\nabla f^T \nabla f$
Second Fundamental Form

Problems:

- The first fundamental form measures length changes only.
- A cylinder looks like a flat sheet in this view.
- We need a tool to measure curvature of a surface as well.
- This requires second order information.
  - Any first order approximation is inherently “flat”.
Second Fundamental Form

Definition:

• Given: regular parametric patch \( f: \mathbb{R}^2 \supseteq \Omega \rightarrow \mathbb{R}^3 \).

• Second fundamental form:
(a.k.a. shape operator, curvature tensor)

\[
S(x_0) = \begin{pmatrix}
\partial_{uu} f(x_0) \cdot n & \partial_{uv} f(x_0) \cdot n \\
\partial_{uv} f(x_0) \cdot n & \partial_{vv} f(x_0) \cdot n
\end{pmatrix}
\]

• Notation:

\[
\Pi(x,y) = x^T \begin{pmatrix}
\partial_{uu} f(x_0) \cdot n & \partial_{uv} f(x_0) \cdot n \\
\partial_{uv} f(x_0) \cdot n & \partial_{vv} f(x_0) \cdot n
\end{pmatrix} y
\]
Second Fundamental Form

Basic Idea:

- Compute second derivative vectors
- Project in normal direction (remove tangential acceleration)
Alternative Formulation (Gauss):

- Local height field parameterization $f(x,y) = z$
- Orthonormal $x,y$ coordinates \textit{tangential} to surface, $z$ in normal direction, origin at zero
- 2nd order Taylor representation:

$$f(x) \approx \frac{1}{2} x^T f''(x)x + f'(x)x + f(0)$$

$$= ex^2 + 2fxy + gy^2$$

- Second fundamental form: Matrix of second derivatives

$$\begin{pmatrix} \partial_{xx}f & \partial_{xy}f \\ \partial_{xy}f & \partial_{yy}f \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$$
Basic Idea

In other words:

- **First fundamental form:** \( I \)
  
  Linear part (squared) of local Taylor approximation.

- **Second fundamental form:** \( II \)
  
  Quadratic part of heightfield approximation

- Both matrices are symmetric.
  
  - Next: eigenanalysis, of course...
Principal Curvature

Eigenanalysis:

• Eigenvalues of second fundamental form for an orthonormal tangent basis are called principal curvatures $\kappa_1, \kappa_2$.

• Corresponding orthogonal eigenvectors are called principal directions of curvature.

$\kappa_i > 0$  $\kappa_0 > 0, \kappa_1 < 0$  $\kappa_0 = 0, \kappa_1 > 0$  $\kappa_0 = 0, \kappa_1 = 0$
Normal Curvature

Definition:

- The *normal curvature* $k(r)$ in direction $r$ for a unit length direction vector $r$ at parameter position $x_0$ is given by:

$$k_{x_0}(r) = II_{x_0}(r,r) = r^T S(x_0) r$$

Relation to Curvature of Plane Curves:

- Intersect the surface locally with plane spanned by normal and $r$ through point $x_0$.
- Identical curvatures (up to sign).
Relation to principal curvature:

- The maximum principal curvature $\kappa_1$ is the maximum of the normal curvature
- The minimum principal curvature $\kappa_2$ is the minimum of the normal curvature
Gaussian & Mean Curvature

More Definitions:
- The Gaussian curvature $K$ is the product of the principal curvatures: $K = \kappa_1 \kappa_2$
- The mean curvature $H$ is the average: $H = 0.5 \cdot (\kappa_1 + \kappa_2)$

Theorems:
- $K(x_0) = \det(S(x_0)) = \frac{eg - f^2}{EG - F^2}$
- $H(x_0) = \frac{1}{2} \text{tr}(S(x_0)) = \frac{eG - 2fF + gE}{2(EG - F^2)}$
Global Properties

Definition:

- An \textit{isometry} is a mapping between surfaces that preserves distances on the surface (geodesics).
- A \textit{developable surface} is a surface with Gaussian curvature zero everywhere (i.e. no curvature in at least one direction).
  - Examples: Cylinder, Cone, Plane
- A developable surface can be locally mapped to a plane isometrically (flattening out, unroll).
Theorema Egregium

Theorema egregium (Gauss):

- Any isometric mapping preserves Gaussian curvature, i.e. Gaussian curvature is invariant under isometric maps ("intrinsic surface property")
- Consequence: The earth ($\approx$ sphere) cannot be mapped to a plane in an exactly length preserving way.
Gauss Bonnet Theorem:

For a compact, orientable surface without boundary in $\mathbb{R}^3$, the area integral of the Gauss curvature is related to the genus $g$ of the surface:

$$\int_S K(x) dx = 4\pi(1 - g)$$

$g = 0$

$g = 1$

$g = 2$

...
Theorem:

• Given two parametric patches in $\mathbb{R}^3$ defined on the same domain $\Omega$.

• Assume that the first and second fundamental form are identical.

• Then there exists a rigid motion that maps one surface to the other.
Summary

Objects are the same up to a rigid motion, if...:

- Curves $\mathbb{R} \rightarrow \mathbb{R}^2$: Same \textit{speed}, same \textit{curvature}
- Curves $\mathbb{R} \rightarrow \mathbb{R}^3$: Same \textit{speed}, same \textit{curvature}, \textit{torsion}
- Surfaces $\mathbb{R}^2 \rightarrow \mathbb{R}^3$: Same \textit{first} & \textit{second} fundamental form
- Volumetric Objects $\mathbb{R}^3 \rightarrow \mathbb{R}^3$: Same \textit{first} fundamental form
Deformation Models

What if this does not hold?

- Deviation in fundamental forms is a measure of deformation

- Example: Surfaces
  - Diagonals of $I_1 - I_2$: scaling (stretching)
  - Off-diagonals of $I_1 - I_2$: sheering
  - Elements of $II_1 - II_2$: bending

- This is the basis of deformation models.