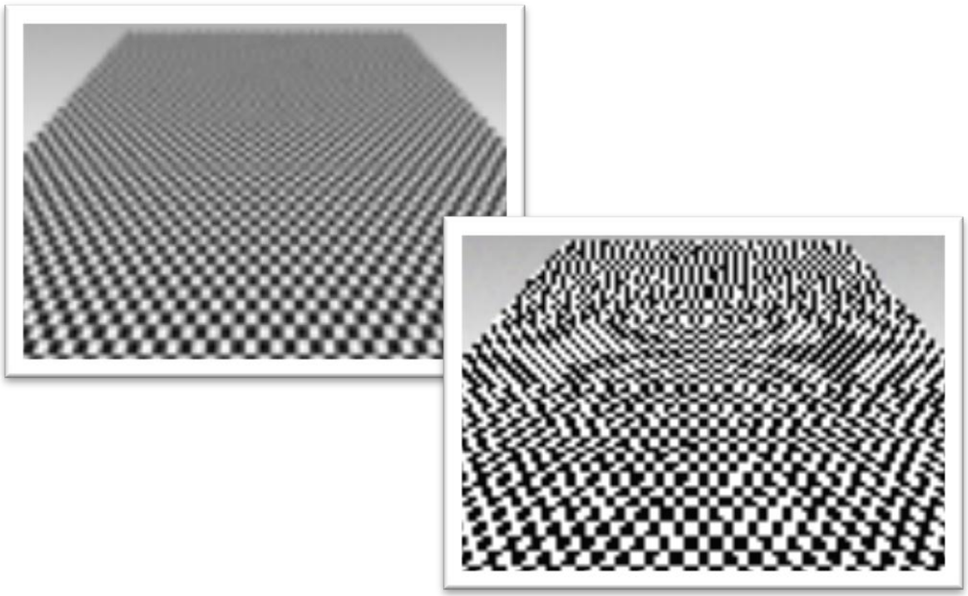
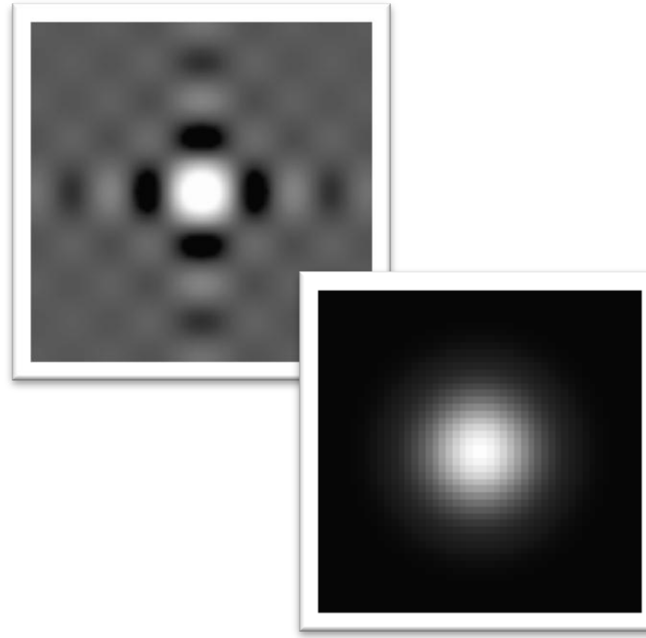
Statistical Geometry Processing

Winter Semester 2011/2012





A Very Short Primer on Signal Theory

UNIVERSITÄT DES SAARLANDES

CLUSTER OF EXCELLENCE

max planck institut informatik

Topics

Topics

- Fourier transform
- Theorems
- Analysis of regularly sampled signals
- Irregular sampling

Fourier Basis

Fourier Basis

- Function space: $\{f: \mathbb{R} \to \mathbb{R}, f \text{ sufficiently smooth}\}$
  - Fourier basis can represent
    - Functions of finite variation
    - Lipchitz-smooth functions
- Basis: sine waves of different *frequency* and *phase*:
  - Real basis:

$$\{\sin 2\pi\omega x, \cos 2\pi\omega x \mid \omega \in \mathbb{R}\}$$

  - Complex variant:

$$\{e^{-2\pi i\omega x} \mid \omega \in \mathbb{R}\}$$

(Euler's formula: $e^{ix} = \cos x + i \sin x$)

Fourier Transform

Fourier Basis properties:

- Fourier basis: $\{e^{-i2\pi\omega x} \mid \omega \in \mathbb{R}\}$
  - Orthogonal basis
  - Projection via scalar products $\Rightarrow$ Fourier transform
- Fourier transform: $(\mathrm{f}: \mathbb{R} \rightarrow \mathbb{C}) \quad \rightarrow \quad (\mathrm{F}: \mathbb{R} \rightarrow \mathbb{C})$

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \omega} dx$$

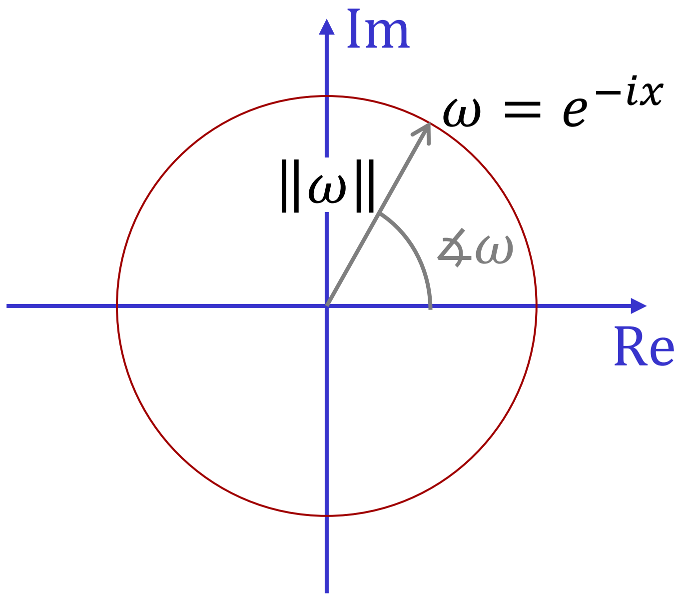
- Inverse Fourier transform: $(\mathrm{F}: \mathbb{R} \rightarrow \mathbb{C}) \rightarrow (\mathrm{f}: \mathbb{R} \rightarrow \mathbb{C})$

$$f(\omega) = \int_{-\infty}^{\infty} F(x) e^{2\pi i x \omega} dx$$

Fourier Transform

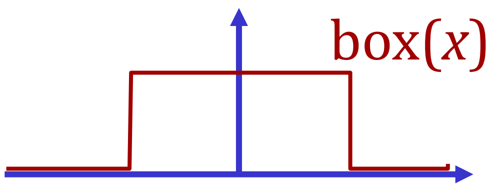
Interpreting the result:

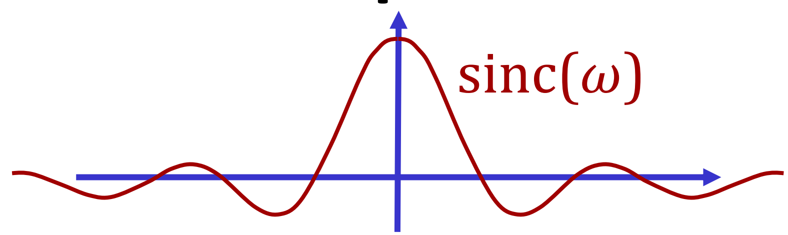
- Transforming a real function f: $\mathbb{R} \rightarrow \mathbb{R}$
- Result: $\mathrm{F}(\omega): \mathbb{R} \rightarrow \mathbb{C}$
  - ω are frequencies (real)
  - Real input f:
    Symmetric $\mathrm{F}(-\omega) = \mathrm{F}(\omega)$
  - Output are complex numbers
    - Magnitude: “power spectrum” (frequency content)
    - Phase: phase spectrum (encodes shifts)



Important Functions

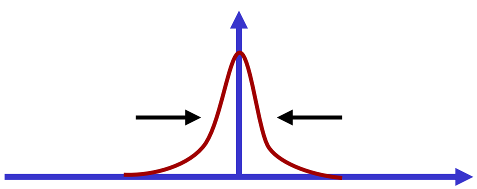
Some important Fourier-transform pairs

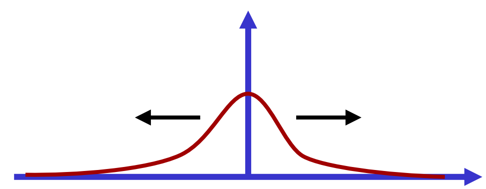




- Box function:

$$f(x) = \mathrm{box}(x) \quad \rightarrow \quad F(\omega) = \frac{\sin\omega}{\omega} := \mathrm{sinc}(\omega)$$





- Gaussian:

$$f(x) = e^{-ax^2} \quad \rightarrow \quad F(\omega) = \sqrt{\frac{\pi}{a}} \cdot e^{-\frac{(\pi\omega)^2}{a}}$$

Higher Dimensional FT

Multi-dimensional Fourier Basis:

- Functions f: $\mathbb{R}^d \rightarrow \mathbb{C}$
- 2D Fourier basis:

$f(x, y)$ represented
as combination of
$$\left\{ e^{-i2\pi\omega_x x} \cdot e^{-i2\pi\omega_y y} \;\middle|\; \omega_x, \omega_y \in \mathbb{R} \right\}$$

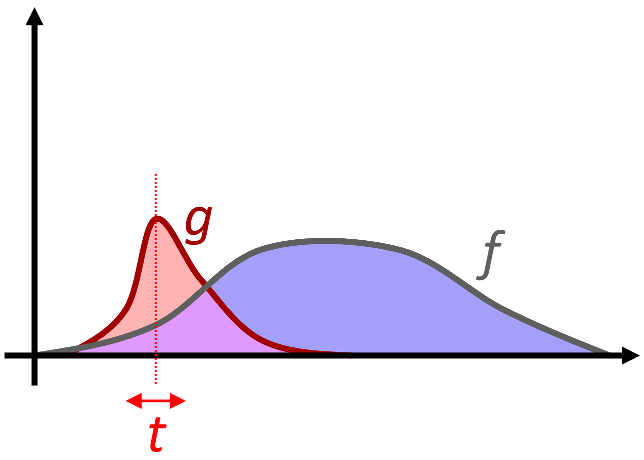
- In general: all combinations of 1D functions

Convolution

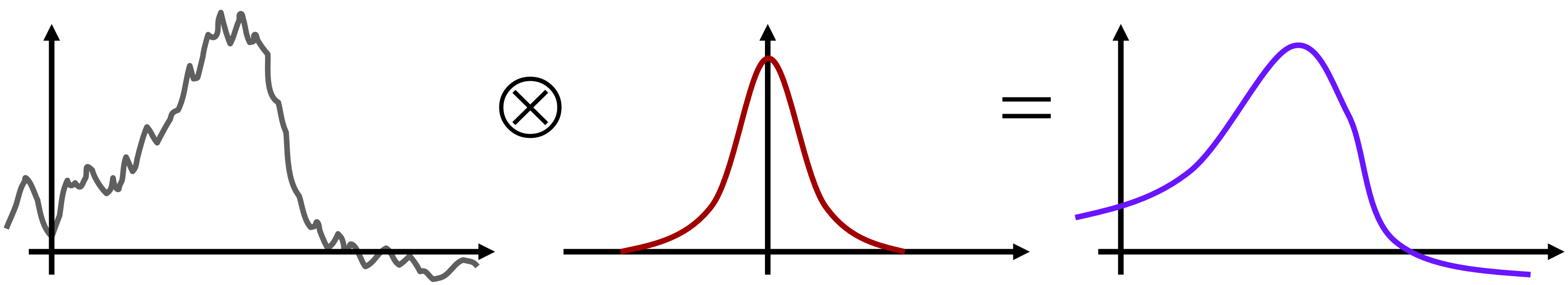
Convolution:

- Weighted average of functions
- Definition:

$$f(t) \otimes g(t) = \int_{-\infty}^{\infty} f(x) g(x-t) dx$$



Example:



Theorems

Fourier transform is an isometry:

- $\langle f, g \rangle = \langle F, G \rangle$
- In particular $\|f\| = \|F\|$

Convolution theorem:

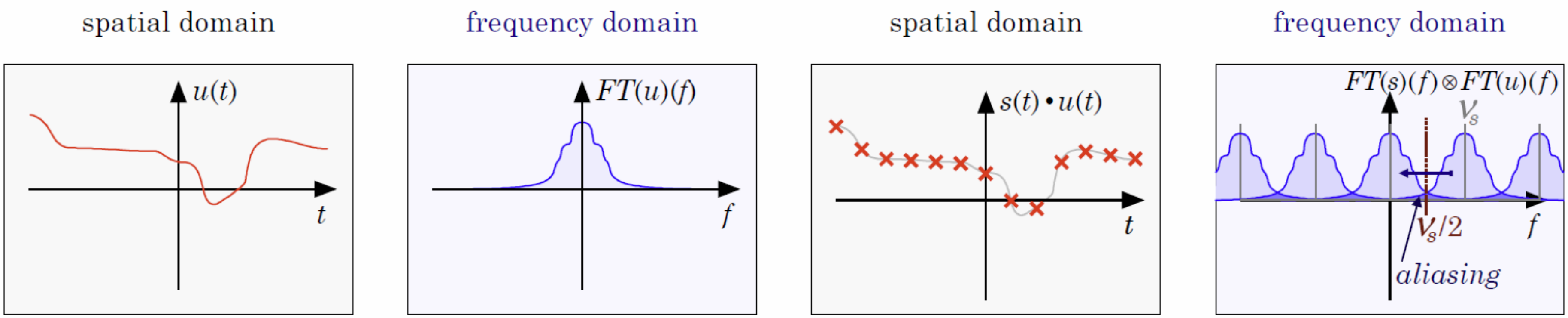
- $FT(f \otimes g) = F \cdot \mathrm{G}$
- Fourier Transform converts convolution into multiplication
    - All other cases as well:
      $FT^{-1}(f \cdot g) = F \otimes \mathrm{G}$, $FT(f \cdot g) = F \otimes \mathrm{G}$, $FT^{-1}(F \cdot G) = F \otimes \mathrm{G}$
    - Fourier basis diagonalizes shift-invariant linear operators

Sampling a Signal

Given:

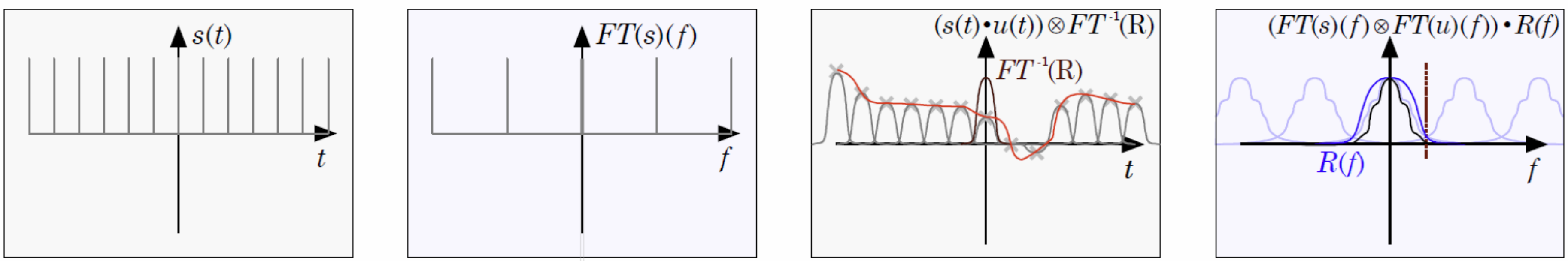
- Signal $f\colon \mathbb{R} \to \mathbb{R}$
- Store digitally:
  - Sample regularly … $f(0.3), f(0.4), f(0.5)$ …
- Question: what information is lost?

Sampling



(a) a continuous function and its frequency spectrum

(c) sampling: frequencies beyond the Nyquest limit $\nu_s/2$ appear as aliasing



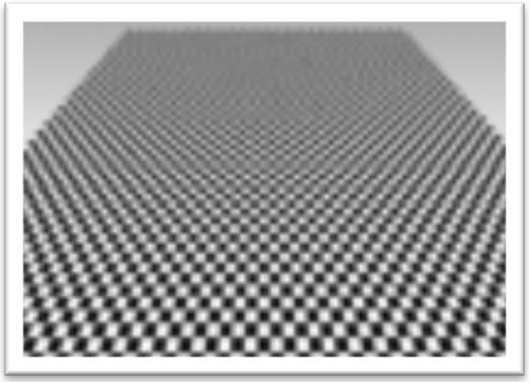
(b) a regular sampling pattern (impulse train) and its frequency spectrum

(d) reconstruction: filtering with a low-pass filter R to remove replicated spectra

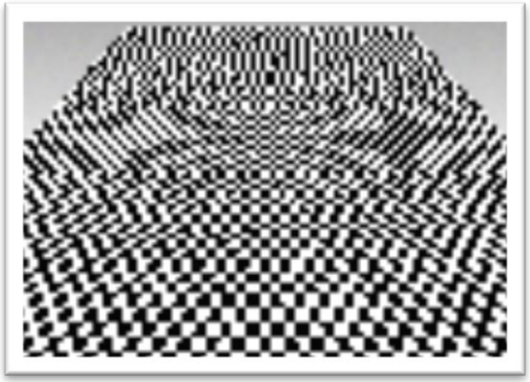
Regular Sampling

Results: Sampling

- Band-limited signals can be represented exactly
  - Sampling with frequency ν_s:
    Highest frequency in Fourier spectrum $\leq \nu_s/2$
- Higher frequencies *alias*
  - Aliasing artifacts (low-frequency patterns)
  - Cannot be removed after sampling (loss of information)



band-limited

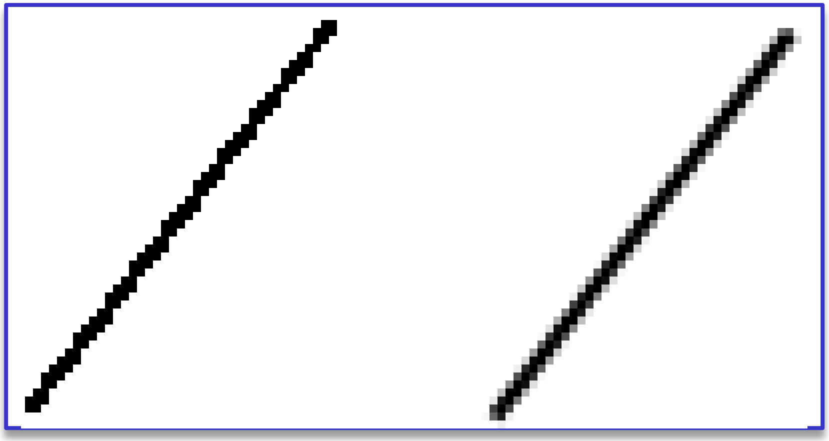


aliasing

Regular Sampling

Result: Reconstruction

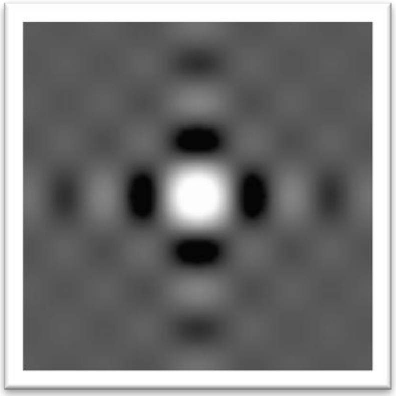
- When reconstructing from discrete samples
- Use band-limited basis functions
  - Highest frequency in Fourier spectrum $\leq \nu_s/2$
  - Otherwise: Reconstruction aliasing



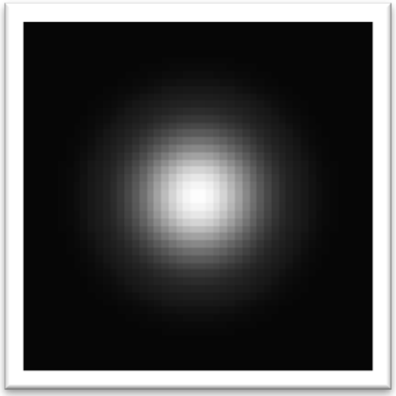
Regular Sampling

Reconstruction Filters

- Optimal filter: sinc
  (no frequencies discarded)
- However:
  - Ringing artifacts in spatial domain
  - Not useful for images (better for audio)
- Compromise
  - Gaussian filter
    (most frequently used)
  - There exist better ones,
    such as Mitchell-Netravalli,
    Lancos, etc...



2D sinc



2D Gaussian

Irregular Sampling

Irregular Sampling

- No comparable formal theory
- However: similar idea
  - Band-limited by “sampling frequency”
  - Sampling frequency = mean sample spacing
    - Not as clearly defined as in regular grids
    - May vary locally (adaptive sampling)
- Aliasing
  - Random sampling creates noise as aliasing artifacts
  - Evenly distributed sample concentrate noise in higher frequency bands in comparison to purely random sampling

Consequences for our applications

When designing bases for function spaces

- Use band-limited functions
- Typical scenario:
  - Regular grid with spacing σ
  - Grid points $\mathbf{g}_i$
  - Use functions: $\exp\left(-\frac{(\mathbf{x}-\mathbf{g}_i)^2}{\sigma^2}\right)$
- Irregular sampling:
  - Same idea
  - Use estimated sample spacing instead of grid width
  - Set σ to average sample spacing to neighbors