Data Mining and Matrices

08 – Boolean Matrix Factorization

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June 13, 2013

Outline

1. Warm-Up
2. What is BMF
3. BMF vs. other three-letter abbreviations
4. Binary matrices, tiles, graphs, and sets
5. Computational Complexity
6. Algorithms
7. Wrap-Up

An example

- Let us consider a data set of people and their traits
  - People: Alice, Bob, and Charles
  - Traits: Long-haired, well-known, and male



long-haired	✓	✓	✗
well-known	✓	✓	✓
male	✗	✓	✓

An example

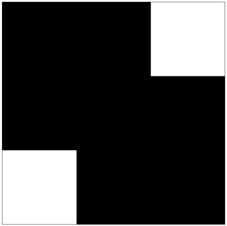
  

long-haired	✓	✓	✗
well-known	✓	✓	✓
male	✗	✓	✓

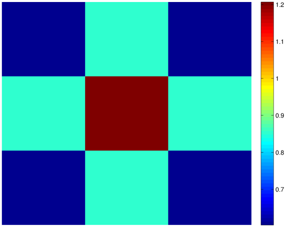
- We can write this data as a binary matrix
- The data obviously has two groups of people and two groups of traits
  -  and  are long-haired and well-known
  -  and  are well-known males
- Can we find these groups automatically (using matrix factorization)?

SVD?

- Could we find the groups using SVD?



The data

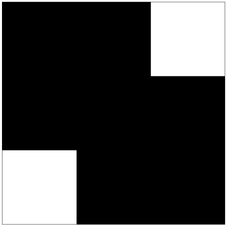


$\mathbf{U}_1 \mathbf{\Sigma}_{1,1} \mathbf{V}_1^T$

- SVD cannot find the groups.

SVD?

- Could we find the groups using SVD?



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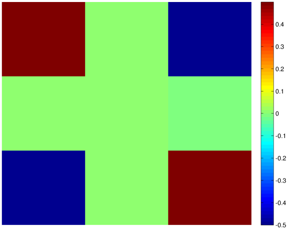
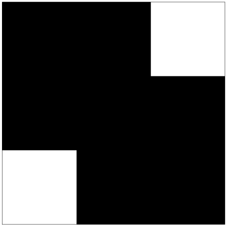


$\mathbf{U}_2\mathbf{\Sigma}_{2,2}\mathbf{V}_2^T$

- SVD cannot find the groups.

SDD?

- The groups are essentially “bumps”, so perhaps SDD?



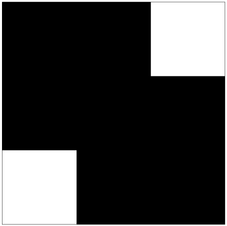
The data

$\mathbf{X}_1 \mathbf{D}_{1,1} \mathbf{Y}_1^T$

- SDD cannot find the groups, either

SDD?

- The groups are essentially “bumps”, so perhaps SDD?



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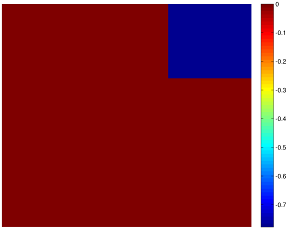

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SDD?

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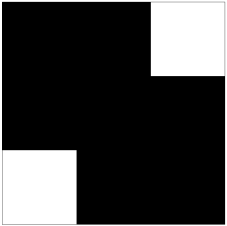

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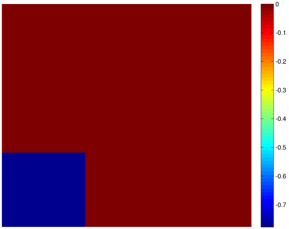
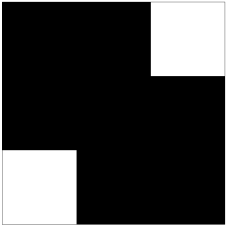


$\mathbf{X}_3\mathbf{D}_{3,3}\mathbf{Y}_3^T$

- SDD cannot find the groups, either

NMF?

- The data is non-negative, so what about NMF?



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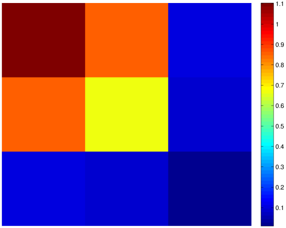
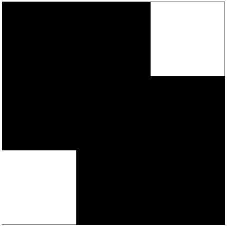


$\mathbf{W}_1\mathbf{H}_1$

- Already closer, but is the middle element in the group or out of the group?

NMF?

- The data is non-negative, so what about NMF?



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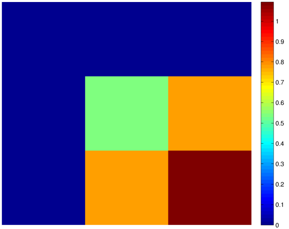
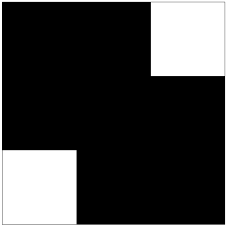


$\mathbf{W}_2\mathbf{H}_2$

- Already closer, but is the middle element in the group or out of the group?

Clustering?

- So NMF's problem was that the results were not precise yes/no. Clustering can do that...



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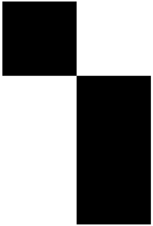
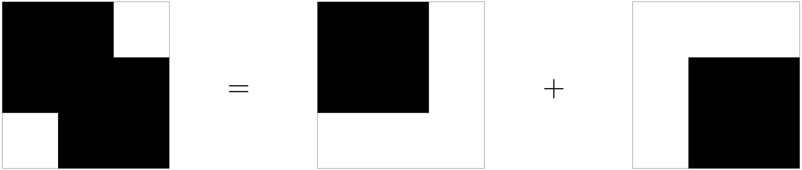


Cluster assignment matrix

- Precise, yes, but arbitrarily assigns  and "well-known" to one of the groups

Boolean matrix factorization

- What we want looks like this:



- The problem: the sum of these two components is *not* the data
  - The center element will have value 2
- Solution: don't care about multiplicity, but let $1 + 1 = 1$

Outline

Boolean matrix product

Boolean matrix product

The **Boolean product** of binary matrices $\mathbf{A} \in \{0,1\}^{m \times k}$ and $\mathbf{B} \in \{0,1\}^{k \times n}$, denoted $\mathbf{A} \boxtimes \mathbf{B}$, is such that

$$(\mathbf{A} \boxtimes \mathbf{B})_{ij} = \bigvee_{\ell=1}^{k} \mathbf{A}_{i\ell}\mathbf{B}_{\ell j} \ .$$

- The matrix product over the *Boolean semi-ring* $(\{0,1\}, \wedge, \vee)$
  - Equivalently, normal matrix product with addition defined as $1 + 1 = 1$
  - Binary matrices equipped with such algebra are called **Boolean matrices**
- The Boolean product is only defined for binary matrices
- $\mathbf{A} \boxtimes \mathbf{B}$ is binary for all $\mathbf{A}$ and $\mathbf{B}$

Definition of the BMF

Boolean Matrix Factorization (BMF)

The (exact) **Boolean matrix factorization** of a binary matrix $\mathbf{A} \in \{0,1\}^{m \times n}$ expresses it as a Boolean product of two factor matrices, $\mathbf{B} \in \{0,1\}^{m \times k}$ and $\mathbf{C} \in \{0,1\}^{k \times n}$. That is $\mathbf{A} = \mathbf{B} \boxtimes \mathbf{C}$.

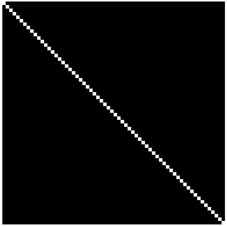
- Typically (in data mining), k is given, and we try to find $\mathbf{B}$ and $\mathbf{C}$ to get as close to $\mathbf{A}$ as possible
- Normally the optimization function is the squared Frobenius norm of the residual, $\|\mathbf{A} - (\mathbf{B} \boxtimes \mathbf{C})\|_F^2$
  - Equivalently, $|\mathbf{A} \oplus (\mathbf{B} \boxtimes \mathbf{C})|$ where
    - $|\mathbf{A}|$ is the sum of values of $\mathbf{A}$ (number of 1s for binary matrices)
    - $\oplus$ is the element-wise exclusive-or (1+1=0)
  - The alternative definition is more "combinatorial" in flavour

The Boolean rank

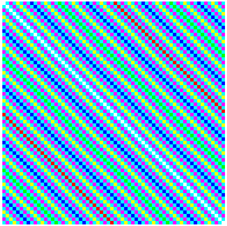
- The **Boolean rank** of a binary matrix $\mathbf{A} \in \{0,1\}^{m \times n}$, $\text{rank}_B(\mathbf{A})$ is the smallest integer k such that there exists $\mathbf{B} \in \{0,1\}^{m \times k}$ and $\mathbf{C} \in \{0,1\}^{k \times n}$ for which $\mathbf{A} = \mathbf{B} \boxtimes \mathbf{C}$
  - Equivalently, the smallest k such that $\mathbf{A}$ is the element-wise *or* of k rank-1 binary matrices
- Exactly like normal or nonnegative rank, but over Boolean algebra
- Recall that for the non-negative rank $\text{rank}_+(\mathbf{A}) \geq \text{rank}(\mathbf{A})$ for all $\mathbf{A}$
- For Boolean and non-negative ranks we have $\text{rank}_+(\mathbf{A}) \geq \text{rank}_B(\mathbf{A})$ for all binary $\mathbf{A}$
  - Essentially because both are anti-negative but BMF can have overlapping components without cost
- Between normal and Boolean rank things are less clear
  - There exists binary matrices for which $\text{rank}(\mathbf{A}) \approx \frac{1}{2}\text{rank}_B(\mathbf{A})$
  - There exists binary matrices for which $\text{rank}_B(\mathbf{A}) = O(\log(\text{rank}(\mathbf{A})))$
  - The logarithmic ratio is essentially the best possible
    - ★ There are at most $2^{\text{rank}_B(\mathbf{A})}$ distinct rows/columns in $\mathbf{A}$

Another example

- Consider the complement of the identity matrix $\bar{\mathbf{I}}$
  - It has full normal rank, but what about the Boolean rank?



$\bar{\mathbf{I}}_{64}$



Boolean rank-12

- The factorization is symmetric on diagonal so we draw two factors at a time
- The Boolean rank of the data is $12 = 2\log_2(64)$

Another example

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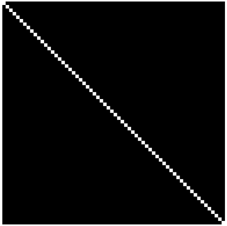

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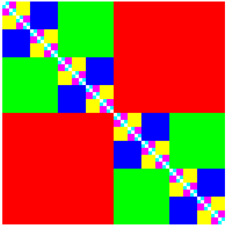
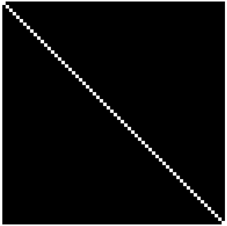


Boolean rank-12

- The factorization is symmetric on diagonal so we draw two factors at a time
- The Boolean rank of the data is $12 = 2\log_2(64)$
- Let's draw the components in reverse order to see the structure

Another example

- Consider the complement of the identity matrix $\overline{\mathbf{I}}$
  - It has full normal rank, but what about the Boolean rank?



$\overline{\mathbf{I}}_{64}$

Factor matrices

- The factorization is symmetric on diagonal so we draw two factors at a time
- The Boolean rank of the data is $12 = 2\log_2(64)$

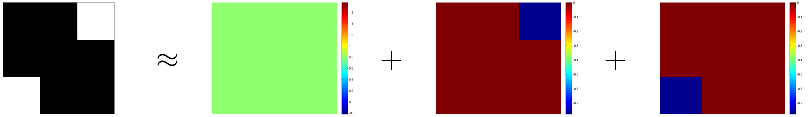
Outline

BMF vs. SVD

- Truncated SVD gives Frobenius-optimal rank-k approximations of the matrix
- But we've already seen that matrices can have smaller Boolean than real rank $\Rightarrow$ BMF can give exact decompositions where SVD cannot
  - Contradiction?
- The answer lies in different algebras: SVD is optimal if you're using the normal algebra
  - BMF can utilize its different addition in some cases very effectively
- In practice, however, SVD usually gives the smallest reconstruction error
  - Even when it's not exactly correct, it's very close
- But reconstruction error isn't all that matters
  - BMF can be more interpretable and more sparse
  - BMF finds different structure than SVD

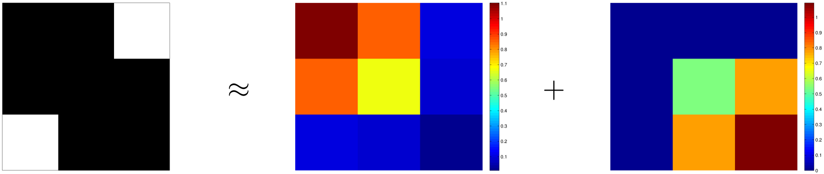
BMF vs. SDD

- Rank-1 binary matrices are sort-of bumps
  - The SDD algorithm can be used to find them
  - But SDD doesn't know about the binary structure of the data
  - And overlapping bumps will cause problems to SDD
- The structure SDD finds is somewhat similar to what BMF finds (from binary matrices)
  - But again, overlapping bumps are handled differently



BMF vs. NMF

- Both BMF and NMF work on anti-negative semi-rings
  - There is no inverse to addition
  - “Parts-of-whole”
- BMF and NMF can be very close to each other
  - Especially after NMF is rounded to binary factor matrices
- But NMF has to scale down overlapping components



BMF vs. clustering

- BMF is a relaxed version of clustering in the hypercube $\{0,1\}^n$
  - The left factor matrix **B** is sort-of cluster assignment matrix, but the "clusters" don't have to partition the rows
  - The right factor matrix **C** gives the centroids in $\{0,1\}^n$
- If we restrict **B** to a cluster assignment matrix (each row has exactly one 1) we get a clustering problem
  - Computationally much easier than BMF
  - Simple local search works well
- But clustering also loses the power of overlapping components

Outline

Frequent itemset mining

- In **frequent itemset mining**, we are given a transaction–item data (who bought what) and we try to find items that are typically bought together
  - A frequent itemset is a set of items that appears in many-enough transactions
- The transaction data can be written as a binary matrix
  - Columns for items, rows for transactions
- Itemsets are subsets of columns
  - Itemset = binary n-dimensional vector $\mathbf{v}$ with $\mathbf{v}_i = 1$ if item i is in the set
- An itemset is frequent if sufficiently many rows have 1s on all columns corresponding to the itemset
  - Let $\mathbf{u} \in \{0,1\}^m$ be such that $\mathbf{u}_j = 1$ iff the itemset is present in transaction j
  - Then $\mathbf{u}\mathbf{v}^T$ is a binary rank-1 matrix corresponding to a **monochromatic** (all-1s) submatrix of the data

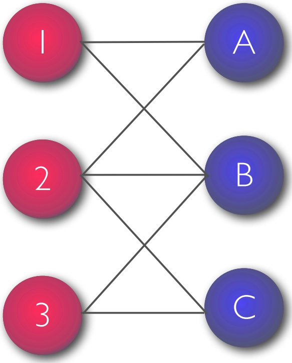
Tiling databases

- When **tiling databases** we try to find **tiles** that cover (most) of the 1s of the data
  - A tile is a monochromatic submatrix of the data (rank-1 binary matrix)
  - A tiling is collection of these tiles such that all (most) 1s of the data belong to at least one of the tiles
- In **minimum tiling**, the goal is to find the least number of tiles such that all 1s in the data belong to at least one tile
- In **maximum k-tiling** the goal is to find k tiles such that as many 1s of the data as possible belong to at least one tile
- In terms of BMF:
  - Tiling with k tiles = rank-k BMF (Boolean sum of k tiles)
  - Tiling can never represent a 0 in the data as a 1
  - Minimum tiling = Boolean rank
  - Maximum k-tiling = best rank-k factorization that never *covers* a 0

Binary matrices and bipartite graphs

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

||

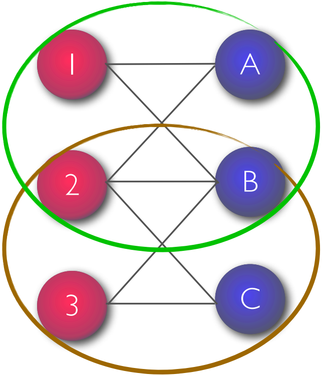


- There is a bijection between $\{0,1\}^{m \times n}$ and (unweighted, undirected) bipartite graphs of $m + n$ vertices
  - Every $\mathbf{A} \in \{0,1\}^{m \times n}$ is a **bi-adjacency matrix** of some bipartite graph $G = (V \cup U, E)$
  - V has m vertices, U has n vertices and $(v_i, u_j) \in E$ iff $\mathbf{A}_{ij} = 1$

BMF and (quasi-)biclique covers

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\|$$

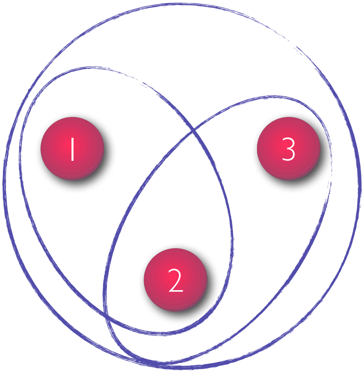


- A **biclique** is a complete bipartite graph
  - Each left-hand-side verted is connected to each right-hand-side vertex
- Each rank-1 binary matrix defines a biclique (subgraph)
  - If $\mathbf{v} \in \{0,1\}^m$ and $\mathbf{u} \in \{0,1\}^n$, then $\mathbf{vu}^T$ is a biclique between $v_i \in V$ and $u_j \in U$ for which $\mathbf{v}_i = \mathbf{u}_j = 1$
- Exact BMF corresponds to covering each edge of the graph with at least one biclique
  - In approximate BMF, quasi-bicliques cover most edges

Binary matrices and sets

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$||$$

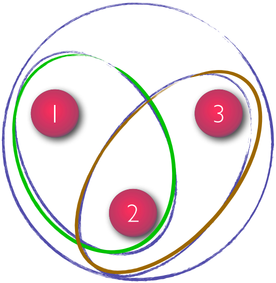


- There is a bijection between $\{0,1\}^{m\times n}$ and sets systems of m sets over n-element universes, $(U, \mathcal{S} \in 2^U)$, $|\mathcal{S}| = m, |U| = n$
  - Up to labeling of elements in U
  - The columns of $\mathbf{A} \in \{0,1\}^{m\times n}$ correspond to the elements of U
  - The rows of $\mathbf{A}$ correspond to the sets in $\mathcal{S}$
  - If $S_i \in \mathcal{S}$, then $u_j \in S_i$ iff $\mathbf{A}_{ij} = 1$

BMF and the Set Basis problem

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\shortparallel$$



- In the **Set Basis** problem, we are given a set system $(U, \mathcal{S})$, and our task is to find collection $\mathcal{C} \subseteq 2^U$ such that we can cover each set $S \in \mathcal{S}$ with a union of some sets of $\mathcal{C}$
  - For each $S \in \mathcal{S}$, there is $\mathcal{C}_S \subseteq \mathcal{C}$ such that $S = \bigcup_{C \in \mathcal{C}_S} C$
- A set basis corresponds to exact BMF
  - The size of the smallest set basis is the Boolean rank
- N.B.: this is the same problem as covering with bicliques

Binary matrices in data mining

- A common use for binary matrices is to represent presence/absence data
  - Animals in spatial areas
  - Items in transactions
- Another common use are binary relations
  - "has seen" between users and movies
  - "links to" between anchor texts and web pages
- Also any directed graphs are typical
- A common problem is that presence/absence data doesn't necessarily tell about absence
  - We know that 1s are probably "true" 1s, but 0s might either be "true" 0s or missing values
    - If a species is not in some area, is it because we haven't seen it or because it's not there?

Outline

The Basis Usage problem

- Alternating projections -style algorithms are very common tool for finding matrix factorizations
  - E.g. the alternating least squares algorithm
- As a subproblem they require you to solve the following problem: Given matrices $\mathbf{Y}$ and $\mathbf{A}$, find matrix $\mathbf{X}$ such that $\|\mathbf{Y} - \mathbf{AX}\|$ is minimized
  - Each column of $\mathbf{X}$ is independent: Given vector $\mathbf{y}$ and matrix $\mathbf{A}$, find a vector $\mathbf{x}$ that minimizes $\|\mathbf{y} - \mathbf{Ax}\|$
    - ★ Linear regression if no constraints on $\mathbf{x}$ and Euclidean norm is used
- The **Basis Usage** problem is the Boolean variant of this problem:

Basis Usage problem

Given binary matrices $\mathbf{A}$ and $\mathbf{B}$, find binary matrix $\mathbf{C}$ that minimizes $\|\mathbf{A} - (\mathbf{B} \boxtimes \mathbf{C})\|_F^2$.

- How hard can it be?

The problem of selecting the best components

- Consider the problem of selecting the best k rank-1 binary matrices from a given set

BMF component selection

Given a binary matrix $\mathbf{A}$, set of n rank-1 binary matrices $S = \{\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n : \text{rank}(\mathbf{S}_i) = 1\}$, and integer k, find $C \subset S$ of size k such that $\|\mathbf{A} - \bigvee_{\mathbf{S} \in C} \mathbf{S}\|_F^2$ is minimized.

- If matrices $\mathbf{S}_i$ are *tiles* of $\mathbf{A}$, this problem is equivalent to the **Max-k cover** problem
  - $\mathbf{S}$ is a tile of $\mathbf{A}$ if for all i, j: when $\mathbf{A}_{ij} = 0$ then $\mathbf{S}_{ij} = 0$
  - The Max k-cover problem: given a set system $(U, \mathcal{S})$, find *partial cover* $\mathcal{C} \subset \mathcal{S}$ of size k ($|\mathcal{C}| = k$) such that $|\bigcup \mathcal{C}| = \left|\bigcup_{C \in \mathcal{C}} C\right|$ is maximized
  - Equivalence: U has an element for each $\mathbf{A}_{ij} = 1$, $S \in \mathcal{S}$ are equivalent to $\mathbf{S} \in S$, and $\bigcup \mathcal{C}$ is equivalent to $\bigvee_{\mathbf{S} \in C} \mathbf{S}$
- But when the matrices $\mathbf{S}_i$ can cover 1s in $\mathbf{A}$, the problem is much harder

The Positive-Negative Partial Set Cover problem

- When the matrices $\mathbf{S}_i$ can cover 1s in $\mathbf{A}$, Max k-cover is not sufficient
  - We need to model the error we make when not covering 1s (as in the Max k-cover)
  - And we need to model the error we make when covering 0s

Positive-Negative Partial Set Cover problem (±PSC)

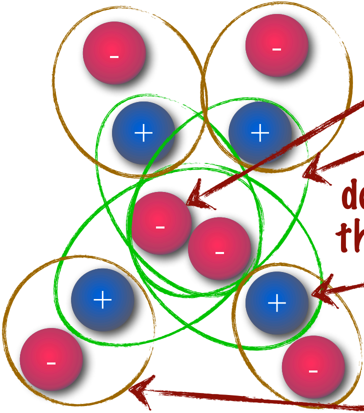
Given a set system $(P \cup N, \mathcal{S} \in 2^{P \cup N})$ and integer k, find a partial cover $\mathcal{C} \subset \mathcal{S}$ of size k such that $\mathcal{C}$ minimizes $|P \setminus (\bigcup \mathcal{C})| + |N \cap (\bigcup \mathcal{C})|$.

- ±PSC minimizes the number of uncovered positive elements plus the number of covered elements
- Equivalence to component selection:
  - Element $\mathbf{A}_{ij} \in P$ if $\mathbf{A}_{ij} = 1$, else $\mathbf{A}_{ij} = N$
  - Each matrix $\mathbf{S}_i \in S$ corresponds to a set S_i in $\mathcal{S}$ ($\mathbf{A}_{ij} \in S_{ell}$ iff $(\mathbf{S}_{ell})_{ij} = 1$
  - $\bigcup \mathcal{C}$ is equivalent to $\bigvee_{\mathbf{S} \in C} \mathbf{S}$
  - $\|\mathbf{A} - \bigvee \mathbf{S}\|_F^2 = |\mathbf{A} \oplus (\bigvee \mathbf{S})|$ (for binary $\mathbf{A}$ and $\mathbf{S}$)

Back to the Basis Usage

- But what has the Basis Usage problem to do with $\pm$PSC?
  - They're also almost equivalent problems
- To see the equivalence, consider the one-column problem: given $\mathbf{a}$ and $\mathbf{B}$, find $\mathbf{c}$ such that $\|\mathbf{a} - \mathbf{Bc}\|_F^2$ is minimized
  - $\mathbf{a}_i \in P$ if $\mathbf{a}_i = 1$, o/w $\mathbf{a}_i \in N$
  - Sets in $\mathcal{S}$ are defined by the columns of $\mathbf{B}$: $\mathbf{a}_i \in S_j$ if $\mathbf{B}_{ij} = 1$
  - If set S_j is selected to $\mathcal{C}$, then $\mathbf{c}_j = 1$ (o/w $\mathbf{c}_j = 0$)
  - And $|P \setminus (\bigcup \mathcal{C})| + |N \cap (\bigcup \mathcal{C})| = |\mathbf{A} \oplus (\mathbf{Bc})| = \|\mathbf{A} - \mathbf{Bc}\|_F^2$
- So while Basis Usage and Component selection look different, they actually are essentially the same problem
  - Unfortunately this is also a hard problem, making algorithm development complicated

Example of $\pm$PSC and Basis Usage



defines the sets

defines the sign

$$
\underset{\boldsymbol{a}}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}
\qquad
\underset{\boldsymbol{B}}{\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}}
$$

Computational complexity

- Computing the Boolean rank is as hard as solving the Set Basis problem, i.e. NP-hard
  - Approximating the Boolean rank is as hard as approximating the minimum chromatic number of a graph, i.e. very hard
  - Compare to normal rank, which is easy save for precision issues
- Finding the least-error approximate BMF is NP-hard
  - And we cannot get any multiplicative approximation factors, as recognizing the case with zero error is also NP-hard
  - The problem is also hard to approximate within additive error
- Solving the $\pm$PSC problem is NP-hard and it is NP-hard to approximate within a superpolylogarithmic factor
  - Therefore, the Basis Usage and Component Selection problems are also NP-hard even to approximate

Outline

Two simple ideas

- **Idea 1:** Alternating updates
  - Start with random **B**, find new **C**, update **B**, etc. until convergence
  - Guaranteed to converge in nm steps for $m \times n$ matrices
  - Problem: requires solving the BU problem
    - But it can be approximated
  - Problem: Converges too fast
    - The optimization landscape is bumpy (many local optima)
- **Idea 2:** Find many dense submatrices (quasi-bicliques) and select from them
  - Existing algorithms find the dense submatrices
  - Finding the dense submatrices is slow
  - Problem: requires solving the BU problem

Expanding tiles: the Panda algorithm

- The Panda algorithm starts by finding large tiles of the matrix
- These are taken one-by-one (from the largest) as a *core* of the next factor
  - The core is expanded by adding to it rows and columns that make it non-monochromatic (add noise)
  - After the extension phase ends, the rows and columns in the expanded core define new component that is added to the factorization
  - The next selected core is the tile that has the largest area **outside** the already-covered area
- Problem: when to stop the extension of the core
  - Panda adds noisy rows and columns to the core as long as that minimizes the noise plus the number of selected rows and columns (poor man's MDL)

Lucchese et al. *Mining Top-k Patterns from Binary Datasets in presence of Noise*. SDM '10

Using association accuracy: the Asso algorithm

- The Asso algorithm uses the correlations between rows to define *candidate factors*, from which it selects the final (column) factors
  - Assume two rows of $\mathbf{A}$ share the same factor
  - Then both of these rows have 1s in the same subset of columns (assuming no noise)
  - Therefore the probability of seeing 1 in the other row on a column we've observed 1 on the other row is high
- Asso computes the empirical probabilities of seeing 1 in row i if it's seen in row j into $m \times m$ matrix
  - This matrix is rounded to binary
  - A greedy search selects a column of this matrix and its corresponding row factor to create the next component
- Problem: requires solving the BU problem
  - Greedy heuristic works well in practice
- Problem: introduces a parameter to round the probabilities
- Problem: noisy or badly overlapping factors do not appear on the rounded matrix

Selecting the parameters: The MDL principle

- Typical matrix factorization methods require the user to pre-specify the rank
  - Also SVD is usually computed only up to some top-k factors
- With BMF, the minimum description length (MDL) principle gives a powerful way to automatically select the rank
- Intuition: data consists of structure and noise
  - Structure can be explained well using the factors
  - Noise cannot be explained well using the factors
- Goal: find the size of the factorization that explains all the structure but doesn't explain the noise
- Idea: Quantify how well we explain the data by how well we can compress it
  - If a component explains many 1s of the data, it's easier to compress the factors than each of the 1s

The MDL principle

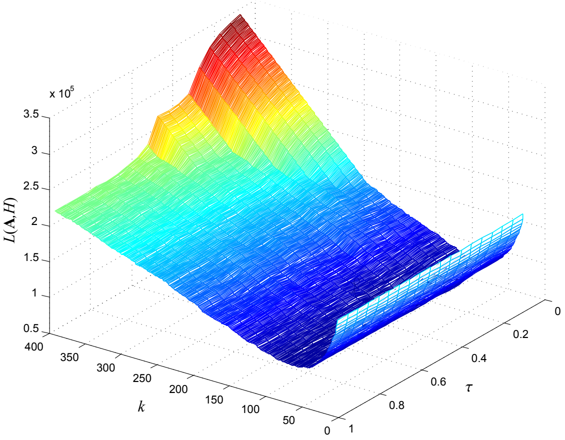
The best rank is the one that lets us to express the data with the least number of bits

MDL for BMF: Specifics

- We compress our data by compressing the factor matrices and the residual matrix
  - The residual is the exclusive or of the data and the factorization, $\mathbf{R} = \mathbf{A} \oplus (\mathbf{B} \boxtimes \mathbf{C})$
  - The residual is needed because the compression must be lossless
- In MDL parlance, **B** and **C** constitute the **hypothesis** and **R** explains the data given the hypothesis
  - Two-part MDL: minimize $L(\mathcal{H}) + L(D \mid \mathcal{H})$, where $L()$ is the encoding length
- Question: how do we encode the matrices?
  - One idea: consider each column of **B** separately
  - Encode the number of 1s in the column, call it b ($\log_2(m)$ bits when m is already known)
  - Enumerate every m-bit binary vector with b 1s in lexicographical order and send the number
    - ★ There are $\binom{m}{b}$ such vectors, so we can encode the number with $\log_2\left(\binom{m}{b}\right)$ bits
    - ★ We don't really need to do the enumeration, just to know how many (fractional) bits it would take

MDL for BMF: An Example

- MDL can be used to find all parameters for the algorithm, not just one
- To use MDL, run the algorithm with different values of k and select the one that gives the smallest description length
  - Usually approximately convex, so no need to try all values of k



Outline

Lessons learned

- BMF finds binary factors for binary data yielding binary approximation
  $\rightarrow$ easier interpretation, different structure than normal algebra
- Many problems associated with BMF are hard even to approximate
  - Boolean rank, minimum-error BMF, Basis Usage, ...
- BMF has very combinatorial flavour
  $\rightarrow$ algorithms are less like other matrix factorization algorithms
- MDL can be used to automatically find the rank of the factorization

Suggested reading

- Slides at `http://www.mpi-inf.mpg.de/~pmiettin/bmf_tutorial/material.html`
- Miettinen et al. *The Discrete Basis Problem*, IEEE Trans. Knowl. Data Eng. 20(10), 2008.
  - Explains the Asso algorithm and the use of BMF (called DBP in the paper) in data mining
- Lucchese et al. *Mining Top-k Patterns from Binary Datasets in presence of Noise*. SDM '10
  - Explains the Panda algorithm
- Miettinen & Vreeken *MDL4BMF: Minimum Description Length for Boolean Matrix Factorization*
  - Explains the use of MDL with BMF