Name: 
Matriculation Number: 
Tutorial Group: A □ B □ C □ D □ E □

<table>
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<tr>
<th>Question</th>
<th>1 (5 Points)</th>
<th>2 (5 Points)</th>
<th>3 (5 Points)</th>
<th>4 (5 Points)</th>
<th>Total (20 points)</th>
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Score: 

General instructions:

- The written test contains 4 questions and is scheduled for 45 minutes. The maximum amount of points you can earn is 20.
- Please verify if your exam consists of 12 pages with 4 questions printed legibly, else contact the examiner immediately.
- No electronic devices (calculator, notebook, tablet, PDA, cell phone) are allowed.
- Answers without sufficient details are void (e.g.: you can’t just say “yes” or “no” as the answer).
- Last page consists of material that you may use to solve the questions. You may detach the last page for your convenience.
- You will be provided additional working sheets if necessary. Make sure to return them along with your solution sheet.
- Please provide your ID card when asked by the examiner.
- Please fill in name, matriculation number (student registration number) and tutor group in the form above and return the solution sheets into the provided box.
- Please sign below.

Student’s Signature

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First Short Test 12 November 2013.
LINEAR ALGEBRA

Problem 1. Consider the following matrix $A$,

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

a) Consider the first and second column of $A$,

$\vec{a}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\vec{a}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

Compute the Euclidean (that is, $L_2$) norm of these two vectors. [2 points]

b) Compute the dot product $\vec{a}_1 \cdot \vec{a}_2$. [1 point]

c) Is $A$ invertible? If yes, give its inverse. If not, explain why. [2 points]

d) Let $I$ be the 2-by-2 identity matrix and let

$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Compute

$$AIAII(A^T)^2IAIAIA^T\vec{v}.$$ [1 point]

Solution
Problem 2. Let $A$ be a 500-by-100 matrix and let $A = U\Sigma V^T$ be its SVD.

a) What are the sizes (numbers of rows and columns) of $U$, $\Sigma$, and $V$? [1 point]

b) How many non-zeros can matrix $\Sigma$ have at most? [1 point]

c) Is matrix $A$ invertible? Explain why/why not. [1 point]

d) Assume that $\text{rank}(A) = 100$ and consider the matrix $\tilde{A}^+ = (A^T A)^{-1} A^T$ (you can assume that the inverse exists). Prove that $\tilde{A}^+$ is the pseudo-inverse of $A$. You can use the fact that $V\Sigma U^T$ is the pseudo-inverse of $A$, that $(XY)^T = Y^T X^T$ for all matrices $X$ and $Y$ for which the product is well-defined, and that $(XY)^{-1} = Y^{-1} X^{-1}$ if $X$ and $Y$ are invertible. [2 points]

Solution
Problem 3. Let $X$ and $Y$ be two discrete random variables such that $X$ takes values from \{1, 2, 3, 4\} and $Y$ takes values from \{3, 6, 12\}. Let their joint mass function $f_{X,Y}$ be as follows:

$$
\begin{array}{c|cccc}
   & 1 & 2 & 3 & 4 \\
\hline
   3 & 1/6 & 0 & 1/12 & 1/12 \\
   6 & 1/6 & 1/12 & 1/12 & 1/12 \\
   12 & 0 & 0 & 1/12 & 1/12 \\
\end{array}
$$

a) What is the marginal distribution of $Y$? [1 point]

b) What is the expected value of $Y$, $E[Y]$? [1 point]

c) What is the conditional expectation of $Y$ given $X$, $E[Y \mid X]$? [2 points]

d) Let $A$ be a random variable with $E[A] = 4$ and let $B$ be a random variable with $E[B] = 6$. Let $C$ be a random variable defined as $2(A + B) - 20$. What is $E[C]$? [1 point]

Solution
STATISTICAL INFERENCE

Problem 4. Suppose we have the following sample of 20 response times from a search engine

\[ X = \{10, 9, 1, 8, 2, 7, 3, 6, 4, 5, 2, 3, 4, 1, 5, 8, 2, 10, 5, 7, 3, 4, 6, 5\} \]

(a) What are the sample mean \( \bar{X} \) and the sample variance \( S^2 \)? [1 point]

(b) What is the 95% confidence interval of \( \bar{X} \) (assuming \( \sigma^2 = 1.44 \))? [2 points]

(c) Is there strong evidence to reject the null hypothesis \( H_0 : \mu = 5.5 \) (assuming \( \hat{s} = 0.25 \))? [2 points]

Solution
ADDITIONAL MATERIAL

Linear algebra

• Identity matrix: $n$-by-$n$ matrix $I$ such that $I_{ij} = 1$ iff $i = j$ and $I_{ij} = 0$ otherwise
• Product with identity matrix: $AI = IA = A$ for all $n$-by-$n$ matrices $A$
• Matrix inverse: $A^{-1}A = AA^{-1} = I$
• Transpose identities: $(A^T)^T = A$ for all $A$; $(AB)^T = B^TA^T$ when the product is well-defined
• Inverse of a product: $(AB)^{-1} = B^{-1}A^{-1}$ if $A$ and $B$ are invertible
• Inverse of orthogonal matrices: $A^T = A^{-1}$ iff $A$ is orthogonal

Probability & Statistics:

• Bayes’ Theorem: $P[A|B] = \frac{P[B|A]P[A]}{P[B]}$
• Law of Total Probability: $P[B] = \sum_{i=1}^{n} P[B|A_i]P[A_i]$ for disjoint events $A_i$ with $\sum_{i}^{n} P[A_i] = 1$
• Expectation: $E[X] = \sum_{k=1}^{\infty} k f_X(k)$ and Variance: $Var[X] = E[X^2] - E[X]^2$ for a discrete RV $X$ with density function $f_X$
• Markov inequality: $P[X \geq t] \leq \frac{E[X]}{t}$ for $t \geq 0$ and a non-neg. RV $X$
• Chebyshev inequality: $P[|X - E[X]| \geq t] \leq \frac{Var[X]}{t^2}$ for $t > 0$ and a non-neg. RV $X$
• Sample Mean: $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ and Sample Variance: $S_X^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$
• For an estimator $\hat{\theta}$ of parameter $\theta$ over i.i.d. samples $\{X_1, X_2, ..., X_i, ..., X_n\}$,
  - If $E[X_i] = \mu$, then $E[\hat{\theta}_n] = \mu$
  - If $Var[X_i] = \sigma^2$, then $Var[\hat{\theta}_n] = \frac{\sigma^2}{n}$
  - Standard Error: $se(\hat{\theta}) = \sqrt{Var[\hat{\theta}_n]}$
  - Mean Squared Error: $MSE[\hat{\theta}_n] = (E[\hat{\theta}_n] - \theta)^2 + Var[\hat{\theta}_n]$
Standard Normal Distribution $\rightarrow z$

Numerical entries represent the probability that a standard normal random variable is between $-\infty$ and $z$ where $z = (x - \mu)/\sigma$.

**Chi-Square Distribution Table**

The shaded area is equal to $\alpha$ for $\chi^2 = \chi^2_{\alpha}$. 

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