



Exploring Properties of Normal Multimodal Logics in Simple Type Theory with LEO-II¹

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Representation Matters

Many proof problems can be effectively solved when

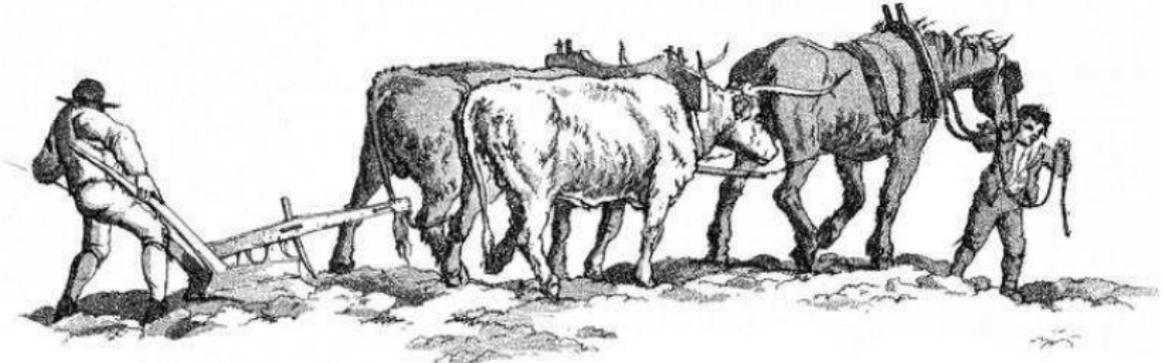
- 1 representing them initially in higher-order logic (expressivity and elegance)
- 2 applying higher-order reasoning techniques to subsequently reduce them to a suitable fragment of higher-order logic
- 3 tackling the reduced problem by an effective specialist reasoner

LEO-II

UNIVERSITY OF
CAMBRIDGE

UNIVERSITÄT
DES
SAARLANDES

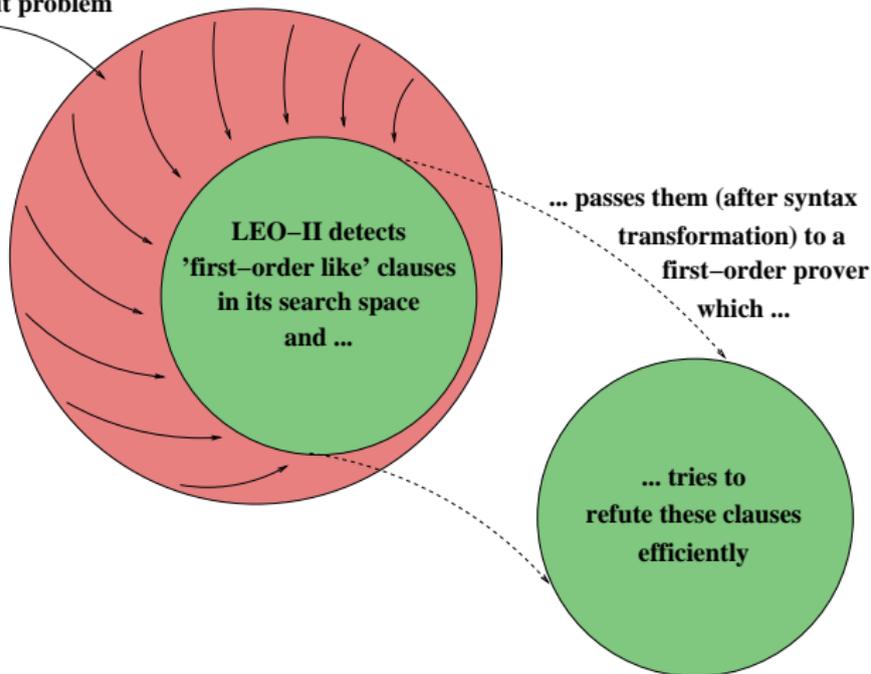
An Effective Higher-Order Theorem Prover



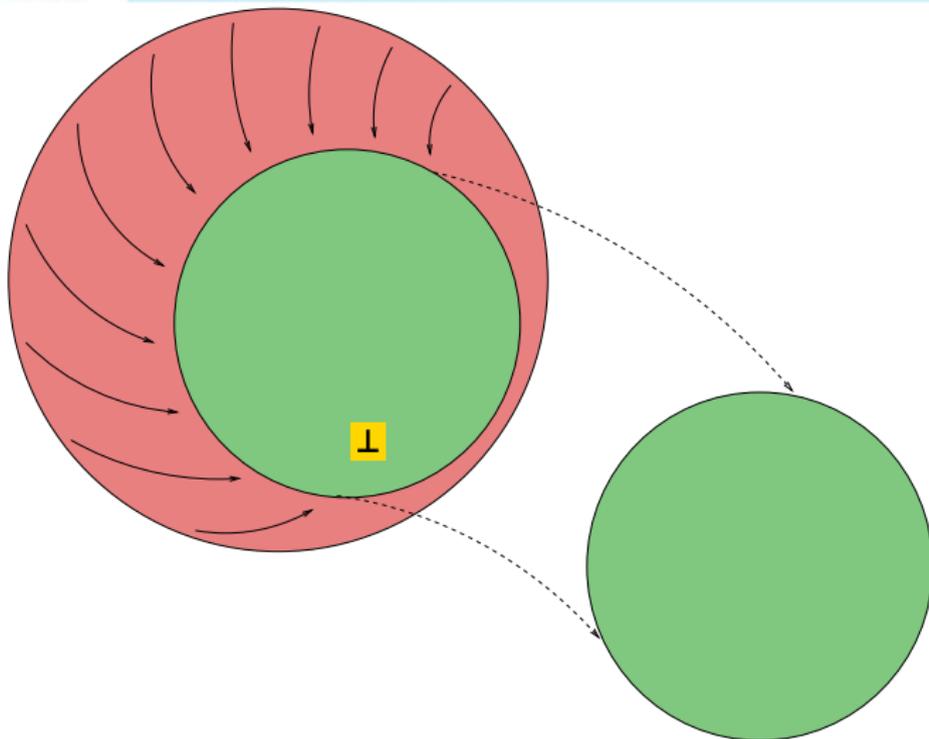
LEO-II employs FO-ATPs:

E, Spass, Vampire

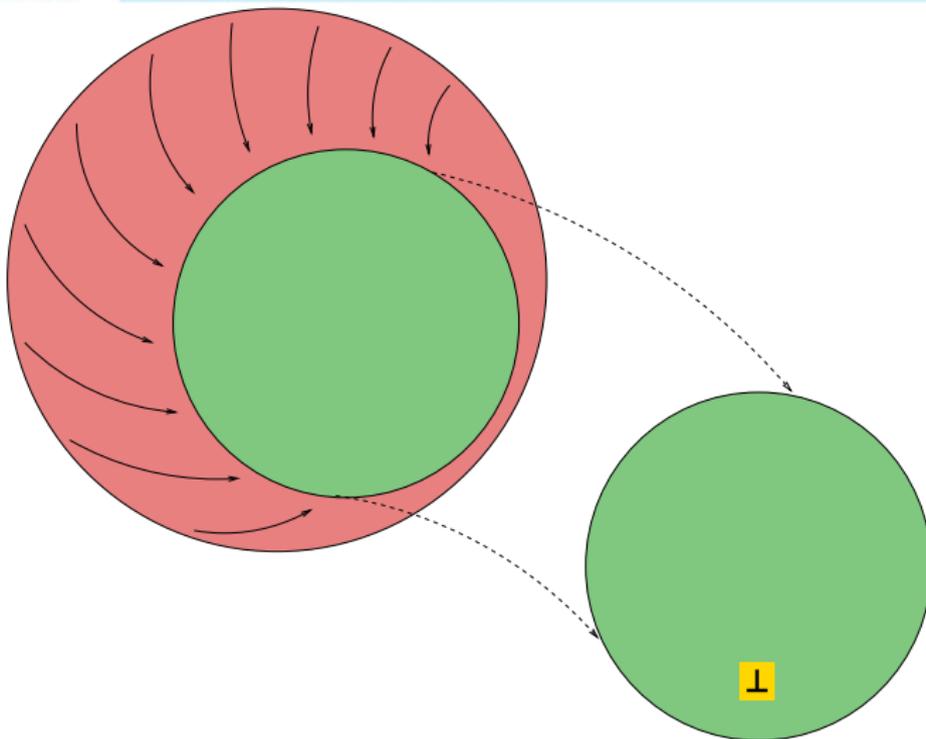
input problem



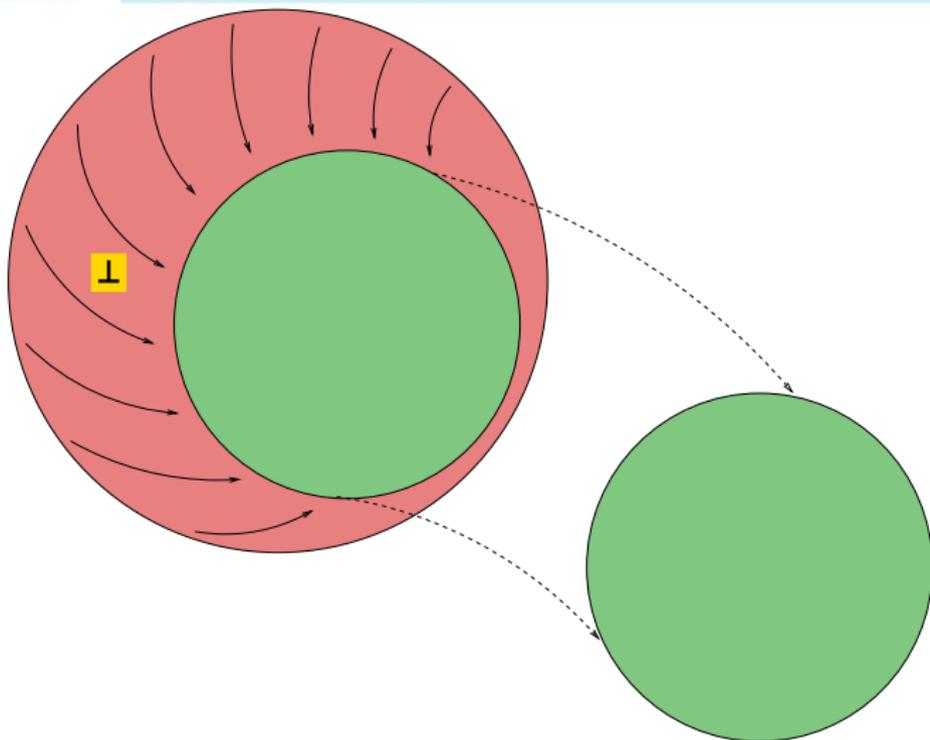
Architecture of LEO-II



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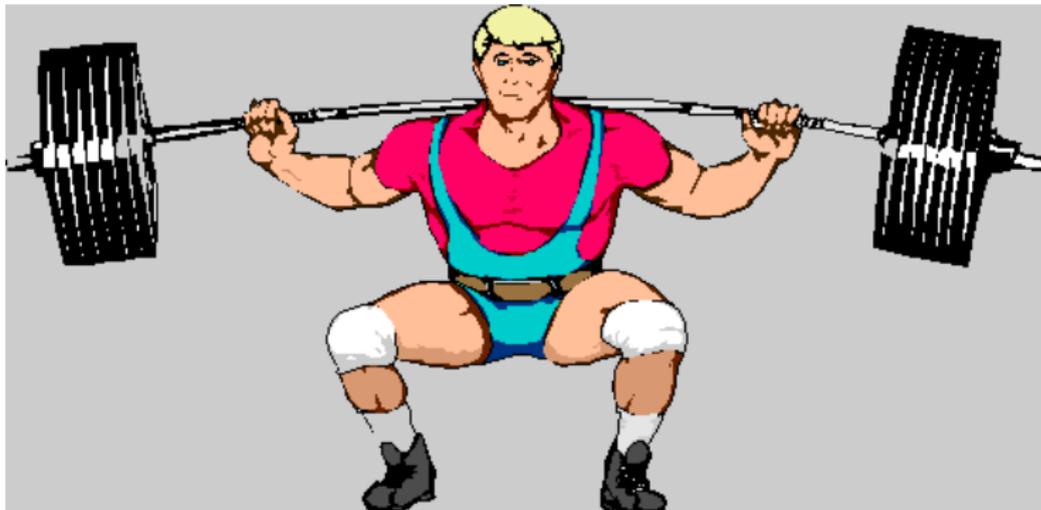
Case Study: Sets, Relations, Functions

Problem	Vamp. 9.0	LEO+Vamp.	LEO-II+E
014+4	114.5	2.60	0.300
017+1	1.0	5.05	0.059
066+1	–	3.73	0.029
067+1	4.6	0.10	0.040
076+1	51.3	0.97	0.031
086+1	0.1	0.01	0.028
096+1	5.9	7.29	0.033
143+3	0.1	0.31	0.034
171+3	68.6	0.38	0.030
580+3	0.0	0.23	0.078
601+3	1.6	1.18	0.089
606+3	0.1	0.27	0.033
607+3	1.2	0.26	0.036
609+3	145.2	0.49	0.039
611+3	0.3	4.00	0.125
612+3	111.9	0.46	0.030
614+3	3.7	0.41	0.060
615+3	103.9	0.47	0.035
623+3	–	2.27	0.282
624+3	3.8	3.29	0.047
630+3	0.1	0.05	0.025
640+3	1.1	0.01	0.033
646+3	84.4	0.01	0.032
647+3	98.2	0.12	0.037

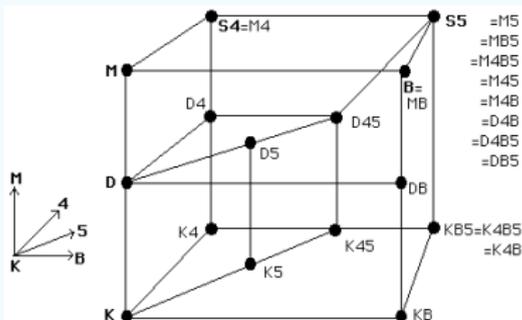
Problem	Vamp. 9.0	LEO+Vamp.	LEO-II+E
648+3	98.2	0.12	0.037
649+3	117.5	0.25	0.037
651+3	117.5	0.09	0.029
657+3	146.6	0.01	0.028
669+3	83.1	0.20	0.041
670+3	–	0.14	0.067
671+3	214.9	0.47	0.038
672+3	–	0.23	0.034
673+3	217.1	0.47	0.042
680+3	146.3	2.38	0.035
683+3	0.3	0.27	0.053
684+3	–	3.39	0.039
716+4	–	0.40	0.033
724+4	–	1.91	0.032
741+4	–	3.70	0.042
747+4	–	1.18	0.028
752+4	–	516.00	0.086
753+4	–	1.64	0.037
764+4	0.1	0.01	0.032

Vamp. 9.0: 2.80GHz, 1GB memory, 600s time limit
LEO+Vamp.: 2.40GHz, 4GB memory, 120s time limit
LEO-II+E: 1.60GHz, 1GB memory, 60s time limit

Solving Less Lightweight Problems



Modal Logics Challenge



John Halleck (U Utah):
<http://www.cc.utah.edu/~nahaj/>
 \$100 Modal Logic Challenge:
www.tptp.org

Example

$$\begin{aligned}
 S4 &= K \\
 + M &: \Box_R A \Rightarrow A \\
 + 4 &: \Box_R A \Rightarrow \Box_R \Box_R A
 \end{aligned}$$

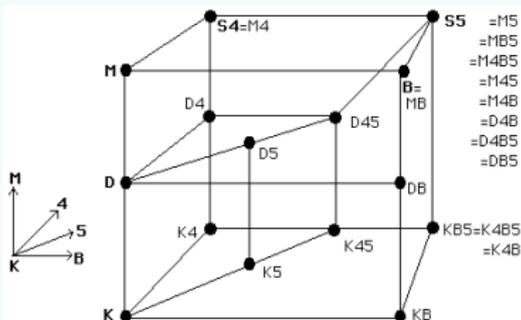
Theorems:

$$\begin{aligned}
 S4 &\not\subseteq K & (1) \\
 (M \wedge 4) &\Leftrightarrow (refl.(R) \wedge trans.(R)) & (2)
 \end{aligned}$$

Experiments

	FO-ATPs [SutcliffeEtal-07]	LEO-II + E [BePa-08]
(1)	16min + 2710s	17.3s
(2)	???	2.4s

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(Normal) Multimodal Logic in HOL

Simple, Straightforward Encoding of Multimodal Logic

- ▶ base type ι : set of possible worlds
- ▶ certain terms of type $\iota \rightarrow o$: multimodal logic formulas
- ▶ multimodal logic operators:

$$\neg_{(\iota \rightarrow o) \rightarrow (\iota \rightarrow o)} = \lambda A_{\iota \rightarrow o}. (\lambda x_{\iota}. \neg A(x))$$

$$\vee_{(\iota \rightarrow o) \rightarrow (\iota \rightarrow o) \rightarrow (\iota \rightarrow o)} = \lambda A_{\iota \rightarrow o}, B_{\iota \rightarrow o}. (\lambda x_{\iota}. A(x) \vee B(x))$$

$$\square_{R_{(\iota \rightarrow \iota \rightarrow o) \rightarrow (\iota \rightarrow o) \rightarrow (\iota \rightarrow o)}} = \lambda R_{\iota \rightarrow \iota \rightarrow o}, A_{\iota \rightarrow o}. (\lambda x_{\iota}. \forall y_{\iota}. R(x, y) \Rightarrow A(y))$$

Related Work

[Gallin-73], [Carpenter-98], [Merz-99],
[\[Brown-05\]](#), [Hardt&Smolka-07], [Kaminski&Smolka-07]

Encoding of Validity

$$\text{valid} := \lambda A_{\iota \rightarrow o}. (\forall w_{\iota}. A(w))$$

Problem	LEO-II + E
$\text{valid}(\Box_r \top)$	0.025s
$\text{valid}(\Box_r a \Rightarrow \Box_r a)$	0.026s
$\text{valid}(\Box_r a \Rightarrow \Box_s a)$	–
$\text{valid}(\Box_s (\Box_r a \Rightarrow \Box_r a))$	0.026s
$\text{valid}(\Box_r (a \wedge b) \Leftrightarrow (\Box_r a \wedge \Box_r b))$	0.044s
$\text{valid}(\Diamond_r (a \Rightarrow b) \Rightarrow \Box_r a \Rightarrow \Diamond_r b)$	0.030s
$\text{valid}(\neg \Diamond_r a \Rightarrow \Box_r (a \Rightarrow b))$	0.029s
$\text{valid}(\Box_r b \Rightarrow \Box_r (a \Rightarrow b))$	0.026s
$\text{valid}((\Diamond_r a \Rightarrow \Box_r b) \Rightarrow \Box_r (a \Rightarrow b))$	0.027s
$\text{valid}((\Diamond_r a \Rightarrow \Box_r b) \Rightarrow (\Box_r a \Rightarrow \Box_r b))$	0.029s
$\text{valid}((\Diamond_r a \Rightarrow \Box_r b) \Rightarrow (\Diamond_r a \Rightarrow \Diamond_r b))$	0.030s

A simple equation between modal logic formulas

$$\forall R. \forall A. \forall B. (\Box_R (A \vee B)) = (\Box_R (B \vee A))$$

- initialisation, definition expansion and normalisation:

$$\begin{aligned} & (\lambda X_L. \forall Y_L. \neg((r X) Y) \vee (a Y) \vee (b Y)) \\ & \neq \\ & (\lambda X_L. \forall Y_L. \neg((r X) Y) \vee (b Y) \vee (a Y)) \end{aligned}$$

A simple equation between modal logic formulas

$$\forall R. \forall A. \forall B. (\Box_R (A \vee B)) = (\Box_R (B \vee A))$$

► functional and Boolean extensionality:

$$\begin{aligned} & \neg((\forall Y_v. \neg((r w) Y) \vee (a Y) \vee (b Y))) \\ & \Leftrightarrow \\ & (\forall Y_v. \neg((r w) Y) \vee (b Y) \vee (a Y)) \end{aligned}$$

Example I

A simple equation between modal logic formulas

$$\forall R. \forall A. \forall B. (\Box_R (A \vee B)) = (\Box_R (B \vee A))$$

► normalisation:

40 : $(bV) \vee (aV) \vee \neg((r w) V) \vee \neg((r w) W) \vee (bW) \vee (aW)$

41 : $((r w) z) \vee ((r w) v)$

42 : $\neg(a z) \vee ((r w) v)$

43 : $\neg(b z) \vee ((r w) v)$

44 : $((r w) z) \vee \neg(a v)$

45 : $\neg(a z) \vee \neg(a v)$

46 : $\neg(b z) \vee \neg(a v)$

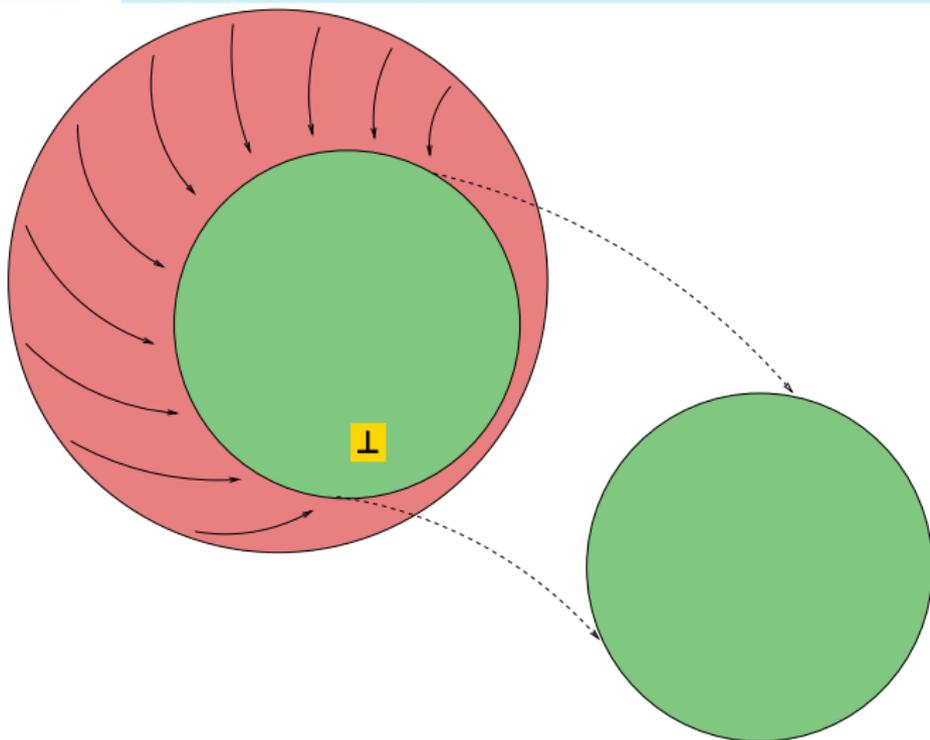
47 : $((r w) z) \vee \neg(b v)$

48 : $\neg(a z) \vee \neg(b v)$

49 : $\neg(b z) \vee \neg(b v)$

► total proving time (notebook with 1.60GHz, 1GB): 0.071s

Architecture of LEO-II



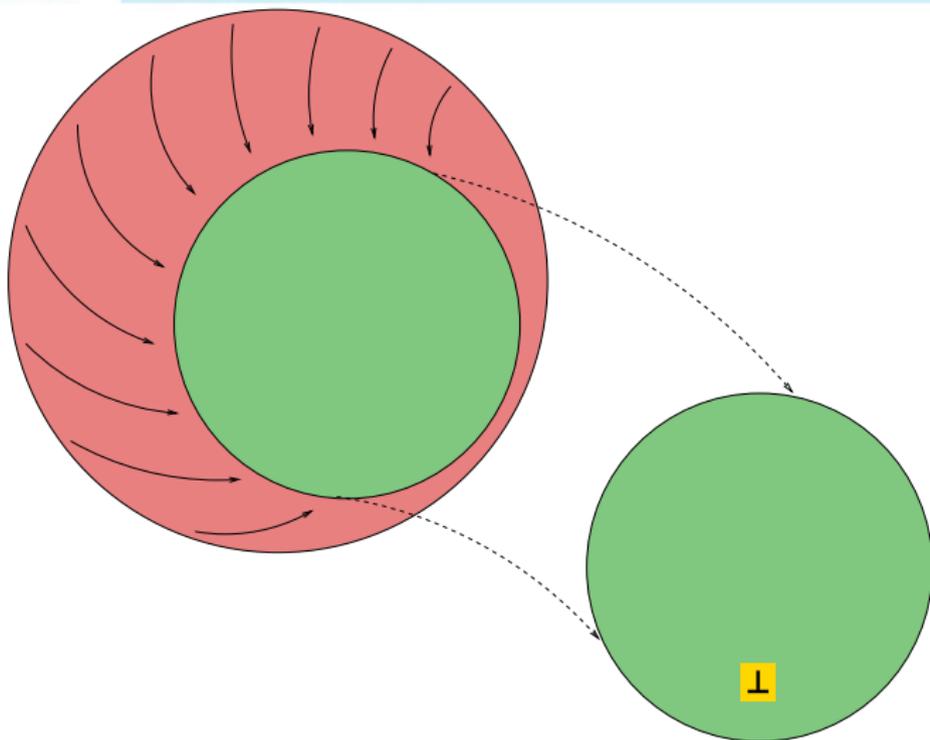
Example II

In modal logic **K**, the axioms *T* and 4 are equivalent to reflexivity and transitivity of the accessibility relation *R*

$$\forall R. (\forall A. \text{valid}(\Box_R A \Rightarrow A) \wedge \text{valid}(\Box_R A \Rightarrow \Box_R \Box_R A)) \\ \Leftrightarrow (\text{reflexive}(R) \wedge \text{transitive}(R))$$

- ▶ processing in LEO-II analogous to previous example
- ▶ now 70 clauses are passed to E
- ▶ E generates **21769** clauses before finding the empty clause
- ▶ total proving time 2.4s
- ▶ this proof cannot be found in LEO-II alone

Architecture of LEO-II

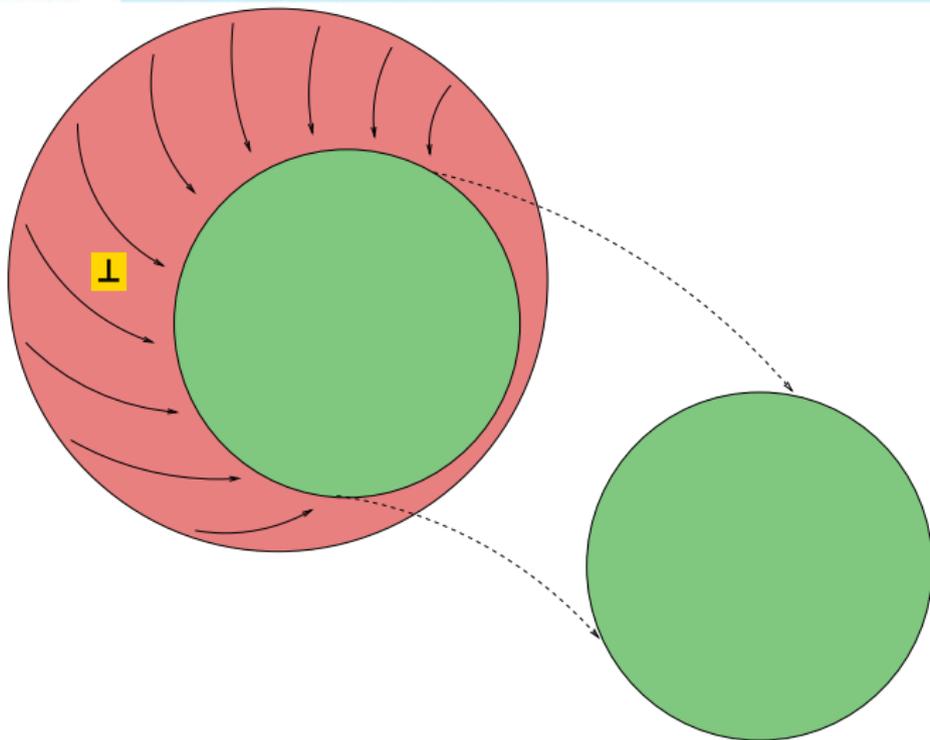


$S4 \not\subseteq K$: Axioms T and 4 are not valid in modal logic K

$$\neg \forall R. \forall A. \forall B. (\text{valid}(\Box_R A \Rightarrow A)) \wedge (\text{valid}(\Box_R B \Rightarrow \Box_R \Box_R B))$$

- ▶ LEO-II shows that axiom T is not valid
- ▶ R is instantiated with \neq via primitive substitution
- ▶ total proving time 17.3s

Architecture of LEO-II



... there is much left to be done!

LEO-II

- ▶ Equational Reasoning
- ▶ Termination
- ▶ Handling of Definitions

Cooperat. with Specialist Reasoners

- ▶ Monadic Second-Order Logic, Prop. Logic, Arithmetic, ...
- ▶ Logic Translations
- ▶ Feedback for LEO-II
- ▶ Proof Transf./Verification
- ▶ Agent-based Architecture

Integration into Proof Assistants

- ▶ Relevance of Axioms
- ▶ Proof Transf./Verification

International Infrastructure

- ▶ TPTP Language(s) for HOL
- ▶ Repository of Proof Problems
- ▶ HOL Prover Contest

Applications

Logic System Interrelationships,
Ontology Reasoning (SUMO, CYC),
Formal Methods, CL, ...



<http://www.cs.miami.edu/~geoff/Conferences/ESHOL/>