# Terminating Tableau Systems for Modal Logic with Equality

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Based on joint work with Mark Kaminski

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**Goal**

Terminating tableau systems for modal logics with equality

**Embedded Approach**

Modal logics as translational fragments of classical logic

Work in progress

# Overview

- Tableaux for pure predicate logic
- Termination for EA
- Equality
- Modal quantifiers
- Safe edges
- Pattern-based termination
- Difference quantifiers
- Transitive modal quantification
- Converse modal quantification

# Tableau Systems

- Can prove that clause is unsatisfiable
- Can prove that clause is finitely satisfiable
- Good for proof search (cut-free sequent system)
- Terminating tableau systems are decision procedures
- Successful for modal logics, description logics
- PL: Beth 1955, Hintikka 1955, Lis 1960, Smullyan 1968
- ML: Kripke 1963, Hughes&Cresswell 1968, Fitting 1972, Pratt 1978
- MLE: Bolander&Braüner 2006, Bolander&Blackburn 2007

# Evidence for Propositional Logic

Formulas in negation normal form

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$F$ evident if

$$\begin{array}{lrcl}
\mathcal{E}_\neg & (\neg s) \in F & \Rightarrow & s \notin F \\
\mathcal{E}_\wedge & (s_1 \wedge s_2) \in F & \Rightarrow & s_1 \in F \wedge s_2 \in F \\
\mathcal{E}_\vee & (s_1 \vee s_2) \in F & \Rightarrow & s_1 \in F \vee s_2 \in F
\end{array}$$

**Theorem (Hintikka 1955)**

*Every evident set is satisfiable.*

# Evidence Conditions Yield Tableau Rules

$$\mathcal{R}_{\neg}\ \frac{\neg s}{\emptyset}\ s \in \Gamma \qquad\qquad \mathcal{R}_{\wedge}\ \frac{s_1 \wedge s_2}{s_1\,,\, s_2} \qquad\qquad \mathcal{R}_{\vee}\ \frac{s_1 \vee s_2}{s_1 \mid s_2}$$

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- $\mathcal{R}_\wedge$ and $\mathcal{R}_\vee$ add witnesses to render conjunctions and disjunctions evident
- Terminating (only subformulas are added)

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$$\frac{\Gamma}{\Gamma_1 \mid \ldots \mid \Gamma_n}$$

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- $\Gamma$ verified: no rule applies
- Semi-completeness: Verified clauses are satisfiable

# Proof Trees

$$
\begin{tikzcd}[arrows=dash]
 & \Gamma \arrow[d] & \\
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\Gamma_2 & & \Gamma_3 \arrow[d] \\
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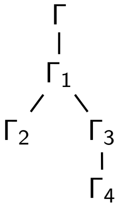


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- Tableaux represent proof trees with sharing
- Proof tree is cut-free sequent derivation ($\Gamma \Rightarrow \emptyset$)

# Pure Predicate Logic

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\begin{array}{rcl}
a & ::= & p x \ldots x \\
s & ::= & a \mid \neg a \mid s \wedge s \mid s \vee s \mid \exists x.s \mid \forall x.s
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$$\exists y.ray \qquad \mathcal{R}_\forall$$

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But $\{\forall x \exists y. rxy,\ pa,\ \exists y. ray,\ raa\}$ is verified

# Termination Proof

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- Ruless add subterms only (modulo instantiation of variables)
- $\mathcal{R}_\exists$ is generative since it adds subformula with new parameter
- Non-generative rules always terminate
- Existential formulas are instantiated only once
- $\mathcal{R}_\exists$ adds only smaller existential formulas

# Nominal Equality and Congruence Closure

$$\begin{array}{rcl} a & ::= & px\ldots x \mid x \doteq x \\ s & ::= & a \mid \neg a \mid s \wedge s \mid s \vee s \mid \exists x.s \mid \forall x.s \end{array}$$

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- Normalizer: $\varphi = \{x := y\}$
- $s \in \tilde{\Gamma} \iff \varphi s \in \varphi \Gamma$
- $\varphi \Gamma$ is basic, i.e., contains only trivial equations $x \doteq x$

# Generalized Rules

Nominal equality does not require new rules, it suffices to generalize $\mathcal{R}_\neg$ and $\mathcal{R}_\exists$

$$\mathcal{R}_\neg \; \frac{\neg s}{\emptyset} \; s \in \tilde{\Gamma} \qquad\qquad \mathcal{R}_\exists \; \frac{\exists x.s}{s^x_y} \; y \notin \mathcal{N}\Gamma \wedge \neg\exists y \colon s^x_y \in \tilde{\Gamma}$$

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## Lemma (Evidence)

*Let $\Gamma$ be verified and $\varphi$ be a normalizer of $\Gamma$.*
*Then $\varphi\Gamma$ is evident.*

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## Lemma (Evidence)

*Let $\Gamma$ be verified and $\varphi$ be a normalizer of $\Gamma$.*
*Then $\varphi\Gamma$ is evident.*

- $\varphi\Gamma$ evident $\Rightarrow$ $\Gamma$ finitely satisfiable
- Results carry over

# Modal Quantifiers

- $\langle r \rangle px = \exists y . rxy \wedge py$ diamond
  at least one $r$-successor of $x$ satisfies $p$

[Hardt&GS HyLo 2006]

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- $\langle r \rangle px = \exists y. rxy \wedge py$ diamond
  at least one $r$-successor of $x$ satisfies $p$
- $[r]px = \forall y. rxy \rightarrow py$ box
  all $r$-successors of $x$ satisfy $p$

[Hardt&GS HyLo 2006]

# Modal Quantifiers

- $\langle r \rangle px = \exists y. rxy \wedge py$ — diamond
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t & ::= & \lambda x.s \mid \langle r \rangle t \mid [r]t
\end{array}
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modal term

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t & ::= & \lambda x.s \mid \langle r \rangle t \mid [r]t
\end{array}
$$

modal term

- PLM translates to PLN with $\beta$-reduction

$$
\begin{array}{lcl}
\langle \_ \rangle & \doteq & \lambda rpx.\ \exists y.\ rxy \wedge py \\
[\_] & \doteq & \lambda rpx.\ \forall y.\ \neg rxy \vee py
\end{array}
$$

# Modal Quantifiers

- $\langle r \rangle px = \exists y.rxy \wedge py$ diamond
- $[r]px = \forall y.rxy \rightarrow py$ box
- Basic modal logic ($t$ closed)

$$
\begin{aligned}
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modal term

# Modal Quantifiers

- $\langle r \rangle px = \exists y.rxy \wedge py$ diamond
- $[r]px = \forall y.rxy \to py$ box
- Basic hybrid logic with global modalities ($t$ closed)

$$
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a & ::= & px \mid x \doteq x \\
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t & ::= & \lambda x.s \mid \langle r \rangle t \mid [r]t
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modal term

# Syntactic Sugar for Modal Terms

$$p \vee \langle r \rangle [r] q$$

Needed for examples and applications but technically redundant

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$$p \vee \langle r\rangle[r]q$$

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$$\lambda x.\ px \vee \langle r \rangle ([r] (\lambda y. qy)) x$$

Needed for examples and applications but technically redundant

# Evidence Conditions for Modal Quantifiers

$$\mathcal{E}_\lambda \quad (\lambda x.s)y \in F \quad \Rightarrow \quad s^x_y \in F$$

# Evidence Conditions for Modal Quantifiers

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$$

$$
\mathcal{R}_\Box\ \frac{[r]sx}{sy}\ x \sim_\Gamma x' \land rx'y \in \Gamma
$$

# Terminating Example

$\langle r \rangle pa, \; [r](a \wedge \langle r \rangle p)a$ initial clause

# Terminating Example

| | |
|---|---|
| $\langle r \rangle pa, \ [r](a \wedge \langle r \rangle p)a$ | initial clause |
| $rab, \ pb$ | $\mathcal{R}_{\Diamond}$ |

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| $a \doteq b,\ \langle r \rangle pb$ | $\mathcal{R}_\wedge$ |

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| $a \doteq b, \; \langle r \rangle pb$ | $\mathcal{R}_\wedge$ |

verified since $rbb \in \tilde{\Gamma}$

# TIT Example

- A relation $r$ is TIT if

| | |
|---|---|
| $\forall x.\ \langle r \rangle \top x$ | totality |
| $\forall x.\ \neg rxx$ | irreflexivity |
| $\forall xyz.\ \neg rxy \vee \neg ryz \vee rxz$ | transitivity |

# TIT Example

- A relation $r$ is TIT if

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- There is no finite relation that is TIT

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- Recall: tableau verifiability implies finite satisfiability

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- $<$ on $\mathbb{N}$ is TIT
- There is no finite relation that is TIT
- Recall: tableau verifiability implies finite satisfiability
- TIT with open modal terms instead of negated edges

$$\begin{array}{cl} \forall x.\ \langle r \rangle \top x & \text{totality} \\ \forall x.\ [r](\neg x)x & \text{irreflexivity} \\ \forall xyz.\ [r](\neg y)x \lor [r](\neg z)y \lor \langle r \rangle zx & \text{transitivity} \end{array}$$

# Simple Formulas

A formula is simple if it does not contain

- subformulas of the form $\neg rxy$ (negated edges)

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- subformulas of the form $\neg rxy$ (negated edges)
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  $\Rightarrow$ tableau rules don't introduce new modal subterms
- existential subterms with non-existentially quantified free variables ($\Rightarrow \mathcal{R}_\exists$ terminates)

# Terminating/Diverging Example

$\langle r \rangle pa, \; [r](\langle r \rangle p)a, \; [r](a \vee a)a$ initial clause

# Terminating/Diverging Example

| | |
|---|---|
| $\langle r \rangle pa,\ [r](\langle r \rangle p)a,\ [r](a \vee a)a$ | initial clause |
| $rab,\ pb,\ \langle r \rangle pb,\ (a \vee a)b$ | $\mathcal{R}_\Diamond, \mathcal{R}_\Box, \mathcal{R}_\Box$ |

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| $a \doteq b \vee a \doteq b$ | $\mathcal{R}_{\lambda}$ |

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| $a \doteq b \vee a \doteq b$ | $\mathcal{R}_\lambda$ |
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verified since $rbb \in \tilde{\Gamma}$

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| $\langle r \rangle pc$ | $\mathcal{R}_\Box\ (rac \in \tilde{\Gamma})$ |

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| $\ldots$ | |

diverges!

# Smart Box Rule for Basic Hybrid Logic

$$\mathcal{R}_\Box \ \frac{[r]sx}{sy} \ x \sim_\Gamma x' \wedge rx'y \in \Gamma$$

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- Exploits that every non-trivial equivalence class contains root (special property of basic hybrid logic)

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- Exploits that every non-trivial equivalence class contains root (special property of basic hybrid logic)
- Yields termination for basic hybrid logic

# Need Safe Edges to Verify Universal Formulas

$pa, \ \forall x. \langle r \rangle px$ initial clause

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| $pa, \ \forall x.\, \langle r \rangle px$ | initial clause |
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| $\langle r \rangle pb$ | $\mathcal{R}_\forall$ |
| $rbb$ | safe edge |

verified!

# Safe Edges and Quasi-Evidence

A safe edge is an edge for which box propagation is already done

$rxy$ safe in $F$ if

- $x, y \in \mathcal{N}F$
- $\neg rxy \notin F$
- $\forall t\colon [r]tx \in F \Rightarrow ty \in F$

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Quasi-Evidence

$\mathcal{E}^q_\Diamond \quad \langle r \rangle sx \in F \;\Rightarrow\; \exists y\colon sy \in F \land rxy$ safe in $F$

# Safe Edges and Quasi-Evidence

A safe edge is an edge for which box propagation is already done

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Quasi-Evidence

$\mathcal{E}^q_\Diamond \quad \langle r\rangle sx \in F \;\Rightarrow\; \exists y\colon sy \in F \wedge rxy$ safe in $F$

## Lemma (Safe Edges)

*If $F$ is quasi-evident, then $F$ together with its safe edges is evident.*

# Pattern-Based Termination

- Pattern: set of modal terms

[Kaminski&GS HyLo 2007]

## Pattern-Based Termination

- Pattern: set of modal terms
- $P$ realized in $F$: $\exists x\ \forall s \in P : sx \in F$

# Pattern-Based Termination

- Pattern: set of modal terms
- $P$ realized in $F$: $\exists x\ \forall s \in P : sx \in F \lor \exists ryx \in F : [r]sy \in F$

# Pattern-Based Termination

- Pattern: set of modal terms
- $P$ realized in $F$: $\exists x\ \forall s \in P: sx \in F \vee \exists ryx \in F: [r]sy \in F$
- $\langle r\rangle sx$ realized in $F$: $\{s\} \cup \{\, t \mid [r]tx \in F \,\}$ realized in $F$

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- $\langle r \rangle sx$ realized in $F$: $\{s\} \cup \{\, t \mid [r]tx \in F \,\}$ realized in $F$
- $\langle r \rangle sx$ realized in $F$ and $F$ satisfies $\mathcal{E}_\Box$ and no negated edges $\Rightarrow \langle r \rangle sx$ quasi-evident in $F$

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$$\mathcal{R}^p_\Diamond \ \frac{\langle r\rangle sx}{rxy,\ sy} \ y \notin \mathcal{N}\Gamma \land \langle r\rangle sx \text{ not realized in } \tilde{\Gamma}$$

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**Theorem** System with $\mathcal{R}^p_\Diamond$ terminates for simple clauses

# Pattern-Based Termination

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Theorem System with $\mathcal{R}^p_\Diamond$ terminates for simple clauses

- $\mathcal{R}^p_\Diamond$ applied to $\langle r\rangle sx$ realizes $\langle r\rangle sx$ in $\tilde{\Gamma}$

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  $\Rightarrow \langle r\rangle sx$ quasi-evident in $F$

$$\mathcal{R}^p_\Diamond\ \frac{\langle r\rangle sx}{rxy,\ sy}\ y \notin \mathcal{N}\Gamma \wedge \langle r\rangle sx \text{ not realized in } \tilde{\Gamma}$$

**Theorem** System with $\mathcal{R}^p_\Diamond$ terminates for simple clauses

- $\mathcal{R}^p_\Diamond$ applied to $\langle r\rangle sx$ realizes $\langle r\rangle sx$ in $\tilde{\Gamma}$
- Realization of patterns is preserved

# Pattern-Based Termination

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- $\langle r\rangle sx$ realized in $F$: $\{s\} \cup \{\, t \mid [r]tx \in F \,\}$ realized in $F$
- $\langle r\rangle sx$ realized in $F$ and $F$ satisfies $\mathcal{E}_\Box$ and no negated edges $\Rightarrow \langle r\rangle sx$ quasi-evident in $F$

$$\mathcal{R}^p_\Diamond\ \frac{\langle r\rangle sx}{rxy,\ sy}\ y \notin \mathcal{N}\Gamma \wedge \langle r\rangle sx \text{ not realized in } \tilde{\Gamma}$$

Theorem System with $\mathcal{R}^p_\Diamond$ terminates for simple clauses

- $\mathcal{R}^p_\Diamond$ applied to $\langle r\rangle sx$ realizes $\langle r\rangle sx$ in $\tilde{\Gamma}$
- Realization of patterns is preserved
- Stock of patterns is finite and preserved

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- Straightforward solution in our framework

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- $\mathcal{R}_D$ adds at most two witnesses per modal subterm $Ds$

[Kaminski&GS M4M 2007]

# Tableau Rules for Difference

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- $\mathcal{R}_D$ adds at most two witnesses per modal subterm $Ds$
- Terminates since D-power is decreased:

$$\begin{aligned} & |\mathrm{Mod}\,\Gamma - \{\, s \mid \exists y\colon sy \in \Gamma \,\}| \\ + \; & |\mathrm{Mod}\,\Gamma - \{\, s \mid \exists x, y\colon \{sx,\; x \dot{\neq} y,\; sy\} \subseteq \Gamma \,\}| \end{aligned}$$

[Kaminski&GS M4M 2007]

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Conflict with addition of safe edges

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[Halpern&Moses 1992]

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$$\mathcal{R}_T^q \ \frac{Tr,\ [r]sx}{[r]sy}\ x \sim_\Gamma x' \wedge rx'y \in \Gamma$$

# Converse Modal Quantifiers

Quantify over predecessors

$$\forall x.\, \langle r \rangle ([r^{-}]p)x$$

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$$\frac{\forall x.\, \langle r \rangle ([r^{-}]p)x, \quad a \doteq a}{\langle r \rangle ([r^{-}]p)a} \; \mathcal{R}_{\forall}$$

# Converse Modal Quantifiers

$$
\begin{array}{ll}
\forall x.\, \langle r \rangle([r^-]p)x, \quad a \doteq a & \\
\langle r \rangle([r^-]p)a & \mathcal{R}_\forall \\
rab, \quad [r^-]pb & \mathcal{R}^q_\Diamond
\end{array}
$$

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rab, \quad [r^-]pb & \mathcal{R}^q_\Diamond \\
pa & \mathcal{R}_\Box
\end{array}$$

# Converse Modal Quantifiers

| | |
|---|---|
| $\forall x.\, \langle r \rangle ([r^-]p)x, \quad a \doteq a$ | |
| $\langle r \rangle ([r^-]p)a$ | $\mathcal{R}_\forall$ |
| $rab, \quad [r^-]pb$ | $\mathcal{R}_\Diamond^q$ |
| $pa$ | $\mathcal{R}_\Box$ |
| $\langle r \rangle ([r^-]p)b$ | $\mathcal{R}_\forall$ |

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- $rbb$ not safe since $pb$ missing

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- $rbb$ now safe, hence $\Gamma$ restricted to $a$, $b$ verified

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pb & \mathcal{R}_\Box \\
\dots &
\end{array}
$$

- *rbb* now safe, hence Γ restricted to *a*, *b* verified
- Still we diverge

# Converse Modal Quantifiers

- With converse quantification pattern-based blocking does not suffice for termination
- Chain-based blocking yields termination [Hughes&Creswell 1968] [Horrocks&Sattler 1999], [Bolander&Blackburn 2007]
- Our equality techniques extend to converse, can do difference with converse for the first time

# Main Contributions

- Use of nominal congruence closure ($\tilde{\Gamma}$)
- Safe edges
- Pattern-based termination
- Termination for $D$
- Termination for transitive relations
- Embedded approach to modal logic

# Method Employed

- Define modal primitives in PLN
- State evidence conditions
- Find quasi-evidence conditions (safe edges)
- State tableau rules (use $\tilde{\Gamma}$)
- Prove evidence lemma ($\varphi\Gamma$ evident)
- Find termination constraints
  - Root propagation for hybrid logic
  - Pattern-based blocking for simple PLM
  - Chain-based blocking for simple PLM with converse

# Conclusions and Outlook

- Equality complicates terminating tableau systems a lot
- Abstract treatment of equality solves many problems
- Embedded approach to modal logic works well
- Work on implementation started
- Vision: $\mu$-calculus and temporal logics with equality