Geometric Quantifier Elimination Heuristic for Octagonal Constraints

Deepak Kapur

Department of Computer Science
University of New Mexico
Albuquerque, NM, USA

with Hengjun Zhao (Chinese Academy of Sciences) and Zhihai Zhang (Peking University)
(work in progress)
• Quantifier Elimination Approach for Generating (Loop) Invariants.
• Quantifier Elimination Approach for Generating (Loop) Invariants.

• Octagonal Constraints.
Outline

- Quantifier Elimination Approach for Generating (Loop) Invariants.
- Octagonal Constraints.
- Geometric and Local Quantifier Elimination Heuristic.
Example

\[
x := 1, \ y := 1, \ z := 0;
\]

while \( x \leq N \) {
    x := x + y + 2;
    y := y + 2;
    z := z + 1
}

return \( z \)
Generating Loop Invariant

- Guess/fix the shape of invariants of interest at various program locations with some parameters which need to be determined.
Generating Loop Invariant

- Guess/fix the shape of invariants of interest at various program locations with some parameters which need to be determined.
- For instance, let

\[
I : \quad A x^2 + B y^2 + C z^2 + D xy + E xz + F yz + G x + H y + J z + K = 0.
\]
Generating Loop Invariant

- Guess/fix the shape of invariants of interest at various program locations with some parameters which need to be determined.
- For instance, let

\[ I : \quad A x^2 + B y^2 + C z^2 + D xy + E xz + F yz + G x + H y + J z + K = 0. \]

- Generate verification conditions using the hypothesized invariants from the code.
Generating Loop Invariant

• Guess/fix the shape of invariants of interest at various program locations with some parameters which need to be determined.
• For instance, let

\[ I : A x^2 + B y^2 + C z^2 + D xy + E xz + F yz + G x + H y + J z + K = 0. \]

• Generate verification conditions using the hypothesized invariants from the code.
  • \textbf{VC1:} At first possible entry of the loop:
    \[ A + B + D + G + H + K = 0. \]
Generating Loop Invariant

- Guess/fix the shape of invariants of interest at various program locations with some parameters which need to be determined.
- For instance, let

\[ I : A x^2 + B y^2 + C z^2 + D \ xy + E \ xz + F \ yz + G \ x + H \ y + J \ z + K = 0. \]

- Generate verification conditions using the hypothesized invariants from the code.
  - **VC1**: At first possible entry of the loop:

\[ A + B + D + G + H + K = 0. \]

  - **VC2**: For every iteration of the loop body:

\[ (I(x, y, z) \land x \leq N) \implies I(x + y + 2, y + 2, z + 1). \]
Generating Loop Invariant

- Guess/fix the shape of invariants of interest at various program locations with some parameters which need to be determined.
- For instance, let

\[ I : \quad A x^2 + B y^2 + C z^2 + D xy + E xz + F yz + G x + H y + J z + K = 0. \]

- Generate verification conditions using the hypothesized invariants from the code.
  - **VC1**: At first possible entry of the loop:
    \[ A + B + D + G + H + K = 0. \]
  - **VC2**: For every iteration of the loop body:
    \[ (I(x, y, z) \land x \leq N) \implies I(x + y + 2, y + 2, z + 1). \]

- Using quantifier elimination, find constraints on parameters \( A, B, C, D, E, F, G, H, J, K \) which ensure that the verification conditions are valid for all possible program variables.
• Constraints on parameters are:

\[ C = -F, \quad J = -2B - F + 2K, \quad G = -4B - F, \quad H = 3B + F - K. \]
• Constraints on parameters are:

\[ C = -F, \quad J = -2B - F + 2K, \quad G = -4B - F, \quad H = 3B + F - K. \]

• Every value of parameters satisfying the above constraints leads to an invariant.
• Constraints on parameters are:

\[ C = -F, \quad J = -2B - F + 2K, \quad G = -4B - F, \quad H = 3B + F - K. \]

• Every value of parameters satisfying the above constraints leads to an invariant.

• 7 parameters and 4 equations, so 3 independent parameters, say \( B, F, K \). Make each to be 1 separately with other independent parameters being 0. Get values of dependent parameters.
• Constraints on parameters are:

\[ C = -F, \quad J = -2B - F + 2K, \quad G = -4B - F, \quad H = 3B + F - K. \]

• Every value of parameters satisfying the above constraints leads to an invariant.
• 7 parameters and 4 equations, so 3 independent parameters, say \( B, F, K \). Make each to be 1 separately with other independent parameters being 0. Get values of dependent parameters.
• Most general invariants describing all invariants are:

\[ y = 2z + 1; \quad z^2 - yz + z + x - y = 0 \quad y^2 - 2z - 4x + 3y = 0, \]
• Constraints on parameters are:

\[ C = -F, \quad J = -2B - F + 2K, \quad G = -4B - F, \quad H = 3B + F - K. \]

• Every value of parameters satisfying the above constraints leads to an invariant.

• 7 parameters and 4 equations, so 3 independent parameters, say \( B, F, K \). Make each to be 1 separately with other independent parameters being 0. Get values of dependent parameters.

• Most general invariants describing all invariants are:

\[
y = 2z + 1; \quad z^2 - yz + z + x - y = 0 \quad y^2 - 2z - 4x + 3y = 0,\]

from which \( x = (z + 1)^2 \) follows.
Method for Automatically Generating Invariants by Quantifier Elimination

- Hypothesize assertions, which are parametrized formulas, at various points in a program.
Method for Automatically Generating Invariants by Quantifier Elimination

- Hypothesize assertions, which are parametrized formulas, at various points in a program.
  - Typically entry of every loop and entry and exit of every procedure suffice.
Method for Automatically Generating Invariants by Quantifier Elimination

- Hypothesize assertions, which are parametrized formulas, at various points in a program.
  - Typically entry of every loop and entry and exit of every procedure suffice.
- Generate verification conditions for every path in the program (a path from an assertion to another assertion including itself).
Method for Automatically Generating Invariants by Quantifier Elimination

- Hypothesize assertions, which are parametrized formulas, at various points in a program.
  - Typically entry of every loop and entry and exit of every procedure suffice.

- Generate verification conditions for every path in the program (a path from an assertion to another assertion including itself).
  - Depending upon the logical language chosen to write invariants, approximations of assignments and test conditions may be necessary.
Method for Automatically Generating Invariants by Quantifier Elimination

- Hypothesize assertions, which are parametrized formulas, at various points in a program.
  - Typically entry of every loop and entry and exit of every procedure suffice.

- Generate verification conditions for every path in the program (a path from an assertion to another assertion including itself).
  - Depending upon the logical language chosen to write invariants, approximations of assignments and test conditions may be necessary.

- Find a formula expressed in terms of parameters eliminating all program variables (using quantifier elimination).
Quality of Invariants

Soundness and Completeness

• Every assignment of parameter values which make the formula true, gives an inductive invariant.
Quality of Invariants

**Soundness and Completeness**

- Every assignment of parameter values which make the formula true, gives an inductive invariant.
  
  - If no parameter values can be found, then invariants of hypothesized forms may not exist. Invariants can be guaranteed not to exist if no approximations are made, while generating verification conditions.
Quality of Invariants

Soundness and Completeness

• Every assignment of parameter values which make the formula true, gives an inductive invariant.
  • If no parameter values can be found, then invariants of hypothesized forms may not exist. Invariants can be guaranteed not to exist if no approximations are made, while generating verification conditions.

• If all assignments making the formula true can be finitely described, invariants generated may be the strongest of the hypothesized form. Invariants generated are guaranteed to be the strongest if no approximations are made, while generating verification conditions.
Domains Admitting Quantifier-Elimination

- Generalized Presburger Arithmetic (for invariants expressed using linear inequalities)
Automatically Generating Invariants

Domains Admitting Quantifier-Elimination

- Generalized Presburger Arithmetic (for invariants expressed using linear inequalities)


- Quantifier Elimination Techniques for Real Closed Fields (REDLOG, QEPCAD)

- Combination of Theories—Presburger Arithmetic with Theory of Equality over Uninterpreted Symbols (Shostak, 1979; Nelson, 1981), and with Boolean Algebra (Kuncak, 2007), etc.

- Reduction Approach to Decision Procedures for Theories over Abstract Data Structures, including Finite Lists, Finite Sets, Finite Arrays, Finite Multisets (Kapur and Zarba, 2005).
Domains Admitting Quantifier-Elimination

- Generalized Presburger Arithmetic (for invariants expressed using linear inequalities)


- Quantifier Elimination Techniques for Real Closed Fields (REDLOG, QEPCAD)
Domains Admitting Quantifier-Elimination

- Generalized Presburger Arithmetic (for invariants expressed using linear inequalities)


- Quantifier Elimination Techniques for Real Closed Fields (REDLOG, QEPCAD)

- Combination of Theories—Presburger Arithmetic with Theory of Equality over Uninterpreted Symbols (Shostak, 1979; Nelson, 1981), and with Boolean Algebra (Kuncak, 2007), etc.
Domains Admitting Quantifier-Elimination

- Generalized Presburger Arithmetic (for invariants expressed using linear inequalities)


- Quantifier Elimination Techniques for Real Closed Fields (REDLOG, QEPCAD)

- Combination of Theories—Presburger Arithmetic with Theory of Equality over Uninterpreted Symbols (Shostak, 1979; Nelson, 1981), and with Boolean Algebra (Kuncak, 2007), etc.

- Reduction Approach to Decision Procedures for Theories over Abstract Data Structures, including Finite Lists, Finite Sets, Finite Arrays, Finite Multisets (Kapur and Zarba, 2005).
Constraint Solving

Constraint Solving

Constraint Solving

- Extensively investigated in many areas including program analysis, program synthesis, termination of programs, as well as hybrid system analysis, particularly safety check and controller synthesis.
Constraint Solving

- Extensively investigated in many areas including program analysis, program synthesis, termination of programs, as well as hybrid system analysis, particularly safety check and controller synthesis.
- Closely related to choosing an abstract domain in the abstract interpretation approach.
How to Scale this Approach

- Quantifier Elimination Methods typically do not scale up due to high complexity.
How to Scale this Approach

- Quantifier Elimination Methods typically do not scale up due to high complexity.
  - Output is huge and difficult to decipher.
How to Scale this Approach

- Quantifier Elimination Methods typically do not scale up due to high complexity.
  - Output is huge and difficult to decipher.
  - In practice, they often do not work (i.e., run out of memory or hang).
How to Scale this Approach

• Quantifier Elimination Methods typically do not scale up due to high complexity.
  • Output is huge and difficult to decipher.
  • In practice, they often do not work (i.e., run out of memory or hang).

• Linear constraint solving on rationals and reals (polyhedral domain), while of polynomial complexity, has been found in practice to be inefficient and slow, especially when used repeatedly as in abstract interpretation approach [Miné]
Octagonal Constraints

- **Octagonal Constraints**: \( l \leq \pm x \pm y \leq h \), a highly restricted subset of linear constraints (at most two variables with coefficients from \( \{-1, 0, 1\} \)).
Octagonal Constraints

- **Octagonal Constraints**: $l \leq \pm x \pm y \leq h$, a highly restricted subset of linear constraints (at most two variables with coefficients from $\{-1, 0, 1\}$).

- This fragment is the most expressive fragment of linear arithmetic over the integers with a polynomial time decision procedure.
Octagonal Constraints

- **Octagonal Constraints**: \( l \leq \pm x \pm y \leq h \), a highly restricted subset of linear constraints (at most two variables with coefficients from \(-1, 0, 1\)).
- This fragment is the most expressive fragment of linear arithmetic over the integers with a polynomial time decision procedure.
- Extending constraints to contain three variables (with just unit coefficients) per inequality makes satisfiability check over the integers NP-complete.
Octagonal Constraints

- **Octagonal Constraints**: \( l \leq \pm x \pm y \leq h \), a highly restricted subset of linear constraints (at most two variables with coefficients from \( \{-1, 0, 1\}\)).
- This fragment is the most expressive fragment of linear arithmetic over the integers with a polynomial time decision procedure.
- Extending constraints to contain three variables (with just unit coefficients) per inequality makes satisfiability check over the integers NP-complete.
- Two variable inequalities with non-unit coefficients over the integers makes the satisfiability check NP-complete.
Octagonal Constraints

- **Octagonal Constraints**: $l \leq \pm x \pm y \leq h$, a highly restricted subset of linear constraints (at most two variables with coefficients from \{-1, 0, 1\}).
- This fragment is the most expressive fragment of linear arithmetic over the integers with a polynomial time decision procedure.
- Extending constraints to contain three variables (with just unit coefficients) per inequality makes satisfiability check over the integers NP-complete.
- Two variable inequalities with non-unit coefficients over the integers makes the satisfiability check NP-complete.
- Class of programs that can be analyzed are very restricted. Still using octagonal constraints (and other heuristics), ASTREE is able to successfully analyze hundreds of thousands of lines of code of numerical software.
Octagonal Constraints

- **Octagonal Constraints**: \( l \leq \pm x \pm y \leq h \), a highly restricted subset of linear constraints (at most two variables with coefficients from \( \{-1, 0, 1\} \)).
- This fragment is the most expressive fragment of linear arithmetic over the integers with a polynomial time decision procedure.
- Extending constraints to contain three variables (with just unit coefficients) per inequality makes satisfiability check over the integers NP-complete.
- Two variable inequalities with non-unit coefficients over the integers makes the satisfiability check NP-complete.
- Class of programs that can be analyzed are very restricted. Still using octagonal constraints (and other heuristics), ASTREE is able to successfully analyze hundreds of thousands of lines of code of numerical software.
- Miné gave well-designed algorithms based on Difference Bound Matrices (DBMs) and graph representation for performing various operations needed for program analysis using the abstract interpretation approach.
Octagonal Constraints

- **Octagonal Constraints**: \( l \leq \pm x \pm y \leq h \), a highly restricted subset of linear constraints (at most two variables with coefficients from \{−1, 0, 1\}).

- This fragment is the most expressive fragment of linear arithmetic over the integers with a polynomial time decision procedure.

- Extending constraints to contain three variables (with just unit coefficients) per inequality makes satisfiability check over the integers NP-complete.

- Two variable inequalities with non-unit coefficients over the integers makes the satisfiability check NP-complete.

- Class of programs that can be analyzed are very restricted. Still using octagonal constraints (and other heuristics), ASTREE is able to successfully analyze hundreds of thousands of lines of code of numerical software.

- Miné gave well-designed algorithms based on Difference Bound Matrices (DBMs) and graph representation for performing various operations needed for program analysis using the abstract interpretation approach.

- Miné’s algorithms are of \( O(n^3) \) (sometimes, \( O(n^4) \)), where \( n \) is the number of variables.
Octagonal Constraints and Quantifier Elimination

- Octagonal constraints have a fixed shape. Given $n$ variables, the most general formula (after simplification) is of the following form:

$$\bigwedge_{i,j} \left( l_{i,j} \leq x_i - x_j \leq b_{i,j}, \ c_{i,j} \leq x_i + x_j \leq d_{i,j}, \ e_i \leq x_i \leq f_i, \ g_j \leq x_j \leq h_j \right)$$

for every pair of variables $x_i, x_j$, where $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}, e_i, f_i, g_j, h_j$ are parameters.
Octagonal Constraints and Quantifier Elimination

- Octagonal constraints have a fixed shape. Given \( n \) variables, the most general formula (after simplification) is of the following form

\[
\bigwedge_{i,j} \left( l_{i,j} : a_{i,j} \leq x_i - x_j \leq b_{i,j}, \quad c_{i,j} \leq x_i + x_j \leq d_{i,j}, \quad e_i \leq x_i \leq f_i \quad g_j \leq x_j \leq h_j \right)
\]

for every pair of variables \( x_i, x_j \), where \( a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}, e_i, f_i, g_j, h_j \) are parameters.

- For a finite program path consisting of a sequence of assignment statements interspersed with tests, its behavior is approximated so that the post condition is also of the above form.
Octagonal Constraints and Quantifier Elimination

- Octagonal constraints have a fixed shape. Given $n$ variables, the most general formula (after simplification) is of the following form
  \[ \wedge_{i,j} \big( l_{i,j} \land a_{i,j} \leq x_i - x_j \leq b_{i,j}, \quad c_{i,j} \leq x_i + x_j \leq d_{i,j}, \quad e_i \leq x_i \leq f_i, \quad g_j \leq x_j \leq h_j \big) \]
  for every pair of variables $x_i, x_j$, where $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}, e_i, f_i, g_j, h_j$ are parameters.

- For a finite program path consisting of a sequence of assignment statements interspersed with tests, its behavior is approximated so that the post condition is also of the above form.

- A verification condition is expressed using atomic formulas that are all octagonal constraints.

  \[ \bigwedge_{i,j} \big( (l_{i,j} \land \alpha(x_i, x_j)) \Rightarrow l'_{i,j} \big), \]

  along with additional parameter-free constraints $\alpha(x_i, x_j)$, of the same form in which lower and upper bounds are constants.
• Analysis of a big conjunctive constraint on every possible pair of variables can be considered individually by considering the subformula on each distinct pair.
Approach: Local QE Heuristics

- Analysis of a big conjunctive constraint on every possible pair of variables can be considered individually by considering the subformula on each distinct pair.

- Consider a precondition, which is a conjunction,
  
  \[ a_{i,j} \leq x_i - x_j \leq b_{i,j}, \quad c_{i,j} \leq x_i + x_j \leq d_{i,j}, \quad e_i \leq x_i \leq f_i, \quad g_j \leq x_j \leq h_j \]

  Assignment statements are of the form \( x_i := x_i + a \) or \( x_i := -x_i + a \). And, tests are lower and upper bounds on variables and expressions of the form \( \pm x_i \pm x_j \). Otherwise, tests and assignments must be approximated.
Approach: Local QE Heuristics

• Analysis of a big conjunctive constraint on every possible pair of variables can be considered individually by considering the subformula on each distinct pair.

• Consider a precondition, which is a conjunction,

\[ a_{i,j} \leq x_i - x_j \leq b_{i,j}, \quad c_{i,j} \leq x_i + x_j \leq d_{i,j}, \quad e_i \leq x_i \leq f_i, \quad g_j \leq x_j \leq h_j \]

Assignment statements are of the form \( x_i := x_i + a \) or \( x_i := -x_i + a \).

And, tests are lower and upper bounds on variables and expressions of the form \( \pm x_i \pm x_j \). Otherwise, tests and assignments must be approximated.

• Quantifier elimination heuristics can be developed using which it is possible to generate constraints on lower and upper bounds by table look ups in \( O(n^2) \) steps, where \( n \) is the number of program variables.
Geometric QE Heuristic

- Analyze how an octagon is affected by transformations due to assignments.
Geometric QE Heuristic

- Analyze how an octagon is affected by transformations due to assignments.
- Identify conditions under which the transformed octagon includes the portion of the original octagon satisfying tests along a program path.
Geometric QE Heuristic

• Analyze how an octagon is affected by transformations due to assignments.
• Identify conditions under which the transformed octagon includes the portion of the original octagon satisfying tests along a program path.
• For each assignment case, a table is built showing the effect on the parameter values by determining the effect on every type of constraints.
Assignments in which signs of variables are reversed

Table 1: Assignments with signs of variables reversed

\[ x_i := -x_i + A, \quad x_j := -x_j + B, \quad \Delta_1 = B - A, \quad \Delta_2 = -A - B, \quad \Delta_3 = -A, \quad \Delta_4 = -B. \]
Assignments in which signs of variables are reversed

Table 1: Assignments with signs of variables reversed

\[ x_i := -x_i + A, \quad x_j := -x_j + B, \quad \Delta_1 = B - A, \quad \Delta_2 = -A - B, \quad \Delta_3 = -A, \quad \Delta_4 = -B. \]

<table>
<thead>
<tr>
<th>Condition</th>
<th>Present</th>
<th>Absent</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_i - x_j \leq a )</td>
<td>( a \leq u_1 )</td>
<td>( u_1 \leq -l_1 - \Delta_1 )</td>
</tr>
<tr>
<td>( x_i - x_j \geq b )</td>
<td>( l_1 \leq b )</td>
<td>( -u_1 - \Delta_1 \leq b )</td>
</tr>
<tr>
<td>( x_i + x_j \leq c )</td>
<td>( c \leq u_2 )</td>
<td>( u_2 \leq -l_2 - \Delta_2 )</td>
</tr>
<tr>
<td>( x_i + x_j \geq d )</td>
<td>( l_2 \leq d )</td>
<td>( -u_2 - \Delta_2 \leq d )</td>
</tr>
<tr>
<td>( x_i \leq e )</td>
<td>( e \leq u_3 )</td>
<td>( u_3 \leq -l_3 - \Delta_3 )</td>
</tr>
<tr>
<td>( x_i \geq f )</td>
<td>( l_3 \leq f )</td>
<td>( -u_3 - \Delta_3 \leq f )</td>
</tr>
<tr>
<td>( x_j \leq g )</td>
<td>( g \leq u_4 )</td>
<td>( u_4 \leq -l_4 - \Delta_4 )</td>
</tr>
<tr>
<td>( x_j \geq h )</td>
<td>( l_4 \leq h )</td>
<td>( -u_4 - \Delta_4 \leq h )</td>
</tr>
</tbody>
</table>
Assignments in which signs of variables do not change

Table 2: No changing signs of variables

\[ x_i := x_i + A, \quad x_j := x_j + B, \quad \Delta_1 = A - B, \quad \Delta_2 = A + B, \quad \Delta_3 = A, \quad \Delta_4 = B. \]
Assignments in which signs of variables do not change

**Table 2: No changing signs of variables**

\[ x_i := x_i + A, \quad x_j := x_j + B, \quad \Delta_1 = A - B, \quad \Delta_2 = A + B, \quad \Delta_3 = A, \quad \Delta_4 = B. \]

<table>
<thead>
<tr>
<th>Condition</th>
<th>Present</th>
<th>Absent</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_i - x_j \leq a ) [\Delta_1 &gt; 0]</td>
<td>( u_1 \geq a + \Delta_1 )</td>
<td>( u_1 = +\infty )</td>
</tr>
<tr>
<td>( x_i - x_j \geq b ) [\Delta_1 &lt; 0]</td>
<td>( l_1 \leq b + \Delta_1 )</td>
<td>( l_1 = -\infty )</td>
</tr>
<tr>
<td>( x_i + x_j \leq c ) [\Delta_2 &gt; 0]</td>
<td>( u_2 \geq c + \Delta_2 )</td>
<td>( u_2 = +\infty )</td>
</tr>
<tr>
<td>( x_i + x_j \geq d ) [\Delta_2 &lt; 0]</td>
<td>( l_2 \leq d + \Delta_2 )</td>
<td>( l_2 = -\infty )</td>
</tr>
<tr>
<td>( x_i \leq e ) [\Delta_3 &gt; 0]</td>
<td>( u_3 \geq e + \Delta_3 )</td>
<td>( u_3 = +\infty )</td>
</tr>
<tr>
<td>( x_i \geq f ) [\Delta_3 &lt; 0]</td>
<td>( l_3 \leq f + \Delta_3 )</td>
<td>( l_3 = -\infty )</td>
</tr>
<tr>
<td>( x_j \leq g ) [\Delta_4 &gt; 0]</td>
<td>( u_4 \geq g + \Delta_4 )</td>
<td>( u_4 = +\infty )</td>
</tr>
<tr>
<td>( x_j \geq h ) [\Delta_4 &lt; 0]</td>
<td>( l_4 \leq h + \Delta_4 )</td>
<td>( l_4 = -\infty )</td>
</tr>
</tbody>
</table>
Assignments in which sign of exactly one variable is changed

Table 3: Sign of exactly one variable is changed

\[ x_i := -x_i + A, \quad x_j := x_j + B, \quad \Delta_1 = B - A, \quad \Delta_2 = -A - B, \quad \Delta_3 = -A, \quad \Delta_4 = B. \]
Assignments in which sign of exactly one variable is changed

Table 3: Sign of exactly one variable is changed

\[ x_i := -x_i + A, \quad x_j := x_j + B, \quad \Delta_1 = B - A, \quad \Delta_2 = -A - B, \quad \Delta_3 = -A, \quad \Delta_4 = B. \]

<table>
<thead>
<tr>
<th>Condition</th>
<th>Present</th>
<th>Absent</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_i - x_j \leq a )</td>
<td>( a \leq u_1 )</td>
<td>( u_1 \leq -l_2 - \Delta_2 )</td>
</tr>
<tr>
<td>( x_i - x_j \geq b )</td>
<td>( l_1 \leq b )</td>
<td>( -u_2 - \Delta_2 \leq b )</td>
</tr>
<tr>
<td>( x_i + x_j \leq c )</td>
<td>( c \leq u_2 )</td>
<td>( u_2 \leq -l_1 - \Delta_1 )</td>
</tr>
<tr>
<td>( x_i + x_j \geq d )</td>
<td>( l_2 \leq d )</td>
<td>( -u_1 - \Delta_1 \leq d )</td>
</tr>
<tr>
<td>( x_i \leq e )</td>
<td>( e \leq u_3 )</td>
<td>( u_3 \leq -l_3 - \Delta_3 )</td>
</tr>
<tr>
<td>( x_i \geq f )</td>
<td>( l_3 \leq f )</td>
<td>( -u_3 - \Delta_3 \leq f )</td>
</tr>
<tr>
<td>( x_j \leq g ) ( \Delta_4 &gt; 0 )</td>
<td>( u_4 \geq g + \Delta_4 )</td>
<td>( u_4 = +\infty )</td>
</tr>
<tr>
<td>( x_j \geq h ) ( \Delta_4 &lt; 0 )</td>
<td>( l_4 \leq h + \Delta_4 )</td>
<td>( l_4 = -\infty )</td>
</tr>
</tbody>
</table>
Assignments in which sign of exactly one variable is changed

Generating Invariants using Table Look-ups

- To determine parameter constraints corresponding to a specific program path, read the corresponding entries from the table.
Assignments in which sign of exactly one variable is changed

Generating Invariants using Table Look-ups

- To determine parameter constraints corresponding to a specific program path, read the corresponding entries from the table.
- Accumulate all such constraints on parameter values. They are also of octagonal form.
Assignments in which sign of exactly one variable is changed

Generating Invariants using Table Look-ups

- To determine parameter constraints corresponding to a specific program path, read the corresponding entries from the table.
- Accumulate all such constraints on parameter values. They are also of octagonal form.
- Every parameter value that satisfies the parameter constraints leads to an invariant.

Maximum values of lower bounds and minimal values of upper bounds satisfying the parameter constraints gives the strongest invariants.

The above can be computed from the frame representation of the octagons.
Assignments in which sign of exactly one variable is changed

Generating Invariants using Table Look-ups

- To determine parameter constraints corresponding to a specific program path, read the corresponding entries from the table.
- Accumulate all such constraints on parameter values. They are also of octagonal form.
- Every parameter value that satisfies the parameter constraints leads to an invariant.
- Maximum values of lower bounds and minimal values of upper bounds satisfying the parameter constraints gives the strongest invariants.
Assignments in which sign of exactly one variable is changed

Generating Invariants using Table Look-ups

- To determine parameter constraints corresponding to a specific program path, read the corresponding entries from the table.
- Accumulate all such constraints on parameter values. They are also of octagonal form.
- Every parameter value that satisfies the parameter constraints leads to an invariant.
- Maximum values of lower bounds and minimal values of upper bounds satisfying the parameter constraints gives the strongest invariants.
- The above can be computed from the frame representation of the octagons.
Assignments in which sign of exactly one variable is changed

A Simple Example

Example

\[
x := 4; y := 6;
\]
while \( x + y \geq 0 \) do
\[
\text{if } (y \geq 6) \text{ then } \{ \ x := -x; \ y := y - 1 \} \]
\[
\text{else } \{ \ x := x - 1; \ y := -y \} \]
endwhile
Assignments in which sign of exactly one variable is changed

A Simple Example

Example

\[
\begin{align*}
\text{x := 4; y := 6;} \\
\text{while (x + y >= 0) do} \\
\quad \text{if (y >= 6) then \{ x := -x; y := y - 1 \}} \\
\quad \text{else \{ x := x - 1; y := -y \}} \\
\text{endwhile}
\end{align*}
\]

\textbf{VC0:} \( I(4, 6) \)

\textbf{VC1:} \( (I(x, y) \land (x + y) \geq 0 \land y \geq 6) \implies I(-x, y - 1). \)

\textbf{VC2:} \( (I(x, y) \land (x + y) \geq 0 \land y < 6) \implies I(x - 1, -y). \)
Assignments in which sign of exactly one variable is changed

**Generating Constraints on Parameters**

- **VC0:** $l_1 \leq -2 \leq u_1 \land l_2 \leq 10 \leq u_2 \land l_3 \leq 4 \geq u_3 \land l_4 \leq 6 \leq u_4$. 

  • Making the $l_i$’s as large as possible and $u_i$’s as small as possible:
    - $l_1 = -10$, $u_1 = 9$,
    - $l_2 = -11$, $u_2 = 10$,
    - $l_3 = -6$, $u_3 = 6$,
    - $l_4 = -5$, $u_4 = 6$.

  • The corresponding invariant is:
    $-10 \leq x - y \leq 9 \land -11 \leq x + y \leq 10 \land -6 \leq x \leq 6 \land -5 \leq y \leq 6$. 

Deepak Kapur (UNM)

Invariants
Assignments in which sign of exactly one variable is changed

Generating Constraints on Parameters

- **VC0**: \( l_1 \leq -2 \leq u_1 \land l_2 \leq 10 \leq u_2 \land l_3 \leq 4 \geq u_3 \land l_4 \leq 6 \leq u_4 \).
- **VC1**: \( x - y: -u_2 - 1 \leq l_1 \land u_1 \leq -l_2 - 1 \).
  \( x + y: l_2 \leq 0 \land -u_1 + 1 \leq 0 \land u_2 \leq -l_1 + 1 \).
  \( x: l_3 + u_3 = 0 \).
  \( y: l_4 \leq 5 \).
Assignments in which sign of exactly one variable is changed

Generating Constraints on Parameters

- **VC0**: \( l_1 \leq -2 \leq u_1 \land l_2 \leq 10 \leq u_2 \land l_3 \leq 4 \geq u_3 \land l_4 \leq 6 \leq u_4 \).

- **VC1**: 
  \( x - y: -u_2 - 1 \leq l_1 \land u_1 \leq -l_2 - 1 \).
  \( x + y: l_2 \leq 0 \land -u_1 + 1 \leq 0 \land u_2 \leq -l_1 + 1 \).
  \( x: l_3 + u_3 = 0 \).
  \( y: l_4 \leq 5 \).

- **VC2**: 
  \( x - y: 10 \leq -l_1 \land -u_2 - 1 \leq -u_1 \land 10 \leq -l_2 - 1 \).
  \( x + y: l_2 \leq 0 \land l_1 + 1 \leq 0 \land u_2 \leq u_1 + 1 \).
  \( x: l_3 \leq -6 \).
  \( y: 5 \leq u_4 \land -u_4 \leq l_4 \land 5 \leq -l_4 \).
Generating Constraints on Parameters

- **VC0**: $l_1 \leq -2 \leq u_1 \land l_2 \leq 10 \leq u_2 \land l_3 \leq 4 \geq u_3 \land l_4 \leq 6 \leq u_4$.
- **VC1**: $x - y$: $-u_2 - 1 \leq l_1 \land u_1 \leq -l_2 - 1$.
  $x + y$: $l_2 \leq 0 \land -u_1 + 1 \leq 0 \land u_2 \leq -l_1 + 1$.
  $x$: $l_3 + u_3 = 0$.
  $y$: $l_4 \leq 5$.
- **VC2**: $x - y$: $10 \leq -l_1 \land -u_2 - 1 \leq -u_1 \land 10 \leq -l_2 - 1$.
  $x + y$: $l_2 \leq 0 \land l_1 + 1 \leq 0 \land u_2 \leq u_1 + 1$.
  $x$: $l_3 \leq -6$.
  $y$: $5 \leq u_4 \land -u_4 \leq l_4 \land 5 \leq -l_4$.
- Making the $l_i$’s as large as possible and $u_i$’s as small as possible:
  $l_1 = -10$, $u_1 = 9$, $l_2 = -11$, $u_2 = 10$, $l_3 = -6$, $u_3 = 6$, $l_4 = -5$, $u_4 = 6$. 
Generating Constraints on Parameters

- **VC0**: \( l_1 \leq -2 \leq u_1 \land l_2 \leq 10 \leq u_2 \land l_3 \leq 4 \geq u_3 \land l_4 \leq 6 \leq u_4 \).
- **VC1**: \( x - y: -u_2 - 1 \leq l_1 \land u_1 \leq -l_2 - 1 \).
  \( x + y: l_2 \leq 0 \land -u_1 + 1 \leq 0 \land u_2 \leq -l_1 + 1 \).
  \( x: l_3 + u_3 = 0 \).
  \( y: l_4 \leq 5 \).
- **VC2**: \( x - y: 10 \leq -l_1 \land -u_2 - 1 \leq -u_1 \land 10 \leq -l_2 - 1 \).
  \( x + y: l_2 \leq 0 \land l_1 + 1 \leq 0 \land u_2 \leq u_1 + 1 \).
  \( x: l_3 \leq -6 \).
  \( y: 5 \leq u_4 \land -u_4 \leq l_4 \land 5 \leq -l_4 \).
- Making the \( l_i \)'s as large as possible and \( u_i \)'s as small as possible:
  \( l_1 = -10, u_1 = 9, l_2 = -11, u_2 = 10, l_3 = -6, u_3 = 6, l_4 = -5, u_4 = 6 \).
- The corresponding invariant is:
  \(-10 \leq x - y \leq 9 \land -11 \leq x + y \leq 10 \land -6 \leq x \leq 6 \land -5 \leq y \leq 6 \).
Local QE Heuristics: Should we propagate?

- Local propagation (i.e., propagate bounds only for the pair of variables appearing in the constraints) on tests to bring them in a canonical form can sometimes improve the bounds. But that is not always clear. The complexity still remains $O(n^2)$.

- Global propagation (meaning propagate bounds from the given constraints on a pair of variables to other constraints in which these variables appear) can sometimes improve the bounds even further. But, then the complexity is $O(n^3)$.

- There are examples for which both local and global propagation (closure) lead to worse results.
Local QE Heuristics: Should we propagate?

- Local propagation (i.e., propagate bounds only for the pair of variables appearing in the constraints) on tests to bring them in a canonical form can sometimes improve the bounds. But that is not always clear. The complexity still remains $O(n^2)$.

- Global propagation (meaning propagate bounds from the given constraints on a pair of variables to other constraints in which these variables appear) can sometimes improve the bounds even further. But, then the complexity is $O(n^3)$. 
Assignments in which sign of exactly one variable is changed

Local QE Heuristics: Should we propagate?

- Local propagation (i.e., propagate bounds only for the pair of variables appearing in the constraints) on tests to bring them in a canonical form can sometimes improve the bounds. But that is not always clear. The complexity still remains $O(n^2)$.

- Global propagation (meaning propagate bounds from the given constraints on a pair of variables to other constraints in which these variables appear) can sometimes improve the bounds even further. But, then the complexity is $O(n^3)$.

- There are examples for which both local and global propagation (closure) lead to worse results.
• QE based methods generate inductive invariants.
Preliminary Comparative Analysis

- QE based methods generate inductive invariants.
- Invariants generated by a complete QE based method are the strongest invariants of the given shape.
Preliminary Comparative Analysis

- QE based methods generate inductive invariants.
- Invariants generated by a complete QE based method are the strongest invariants of the given shape.
- Any method for generating invariants that uses fixed point computation, also generates inductive invariants, and hence, cannot generate stronger invariants than those generated by a complete QE method insofar as these invariants are of the same form.
Preliminary Comparative Analysis

- QE based methods generate inductive invariants.
- Invariants generated by a complete QE based method are the strongest invariants of the given shape.
- Any method for generating invariants that uses fixed point computation, also generates inductive invariants, and hence, cannot generate stronger invariants than those generated by a complete QE method insofar as these invariants are of the same form.
- It has been found in practice that QE based methods generate stronger inductive invariants than methods based on abstract interpretation for polyhedral domain.
Preliminary Comparative Analysis

- QE based methods generate inductive invariants.
- Invariants generated by a complete QE based method are the strongest invariants of the given shape.
- Any method for generating invariants that uses fixed point computation, also generates inductive invariants, and hence, cannot generate stronger invariants than those generated by a complete QE method insofar as these invariants are of the same form.
- It has been found in practice that QE based methods generate stronger inductive invariants than methods based on abstract interpretation for polyhedral domain.
- An open question: the strength of invariants generated by the proposed incomplete heuristics.
Summary

- Quantifier-elimination heuristics might be an alternative to abstract interpretation for program analysis.
• Quantifier-elimination heuristics might be an alternative to abstract interpretation for program analysis.

• Since general (complete) QE methods are very expensive and their outputs are hard to decipher, it is better to consider special cases, sacrificing completeness as well as generality.
Summary

- Quantifier-elimination heuristics might be an alternative to abstract interpretation for program analysis.
- Since general (complete) QE methods are very expensive and their outputs are hard to decipher, it is better to consider special cases, sacrificing completeness as well as generality.
- There is a real trade-off between resources/efficiency and precision/incompleteness.
- Many bells and whistles are needed, just like in the abstract interpretation approach.
• Quantifier-elimination heuristics might be an alternative to abstract interpretation for program analysis.

• Since general (complete) QE methods are very expensive and their outputs are hard to decipher, it is better to consider special cases, sacrificing completeness as well as generality.

• There is a real trade-off between resources/efficiency and precision/incompleteness.

• Many bells and whistles are needed, just like in the abstract interpretation approach.

• An implementation will determine whether the approach is effective.