Cutting to the Chase
Solving Linear Integer Arithmetic

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Deduction at Scale, 2011

Outline

Outline

1 Introduction

2 Cutting to the Chase

3 Experimental Results

Fragments of Integer Satisfiability

Boolean Satisfiability (SAT)

$$x_n \vee \cdots \vee x_1 \vee \neg y_m \vee \cdots \vee \neg y_1$$

Pseudo-Boolean Satisfiability (PBS)

$$a_n x_n + \cdots + a_1 x_1 + a_0 \leq 0 \qquad x_i \in \{0, 1\}$$

Integer Linear Programming (ILP)

$$a_n x_n + \cdots + a_1 x_1 + a_0 \leq 0$$

Boolean Satisfiability

$$\boxed{x_n \vee \cdots \vee x_1 \vee \neg y_m \vee \cdots \vee \neg y_1}$$

- **Resolution-Based procedure** by Martin Davis and Hilary Putnam (1960)
- **Backtracking-Based procedure** by Davis, Putnam, Logemann, Loveland (1962)

Boolean Satisfiability: CDCL

[1996] Marques-Silva, Sakallah
GRASP: A new search algorithm for satisfiabiliy

[2001] Moskewicz, Madigan, Zhao, Zhang, Malik
CHAFF: Engineering an efficient SAT solver

Conflict-Directed Clause Learning

- Use the search to guide resolution
- Use resolution to guide the search

Boolean Satisfiability: CDCL

$$\boxed{x_n \vee \cdots \vee x_1 \vee \neg y_m \vee \cdots \vee \neg y_1}$$

$$\text{RESOLVE} \quad \frac{A \vee x \qquad B \vee \neg x}{A \vee B}$$

Boolean Satisfiability: CDCL

$$\boxed{x_n + \cdots + x_1 + (1 - y_m) + \cdots + (1 - y_1) \geq 1}$$

$$\text{RESOLVE} \quad \frac{A \vee x \qquad B \vee \neg x}{A \vee B}$$

Boolean Satisfiability: CDCL

$$\boxed{x_n + \cdots + x_1 + (1 - y_m) + \cdots + (1 - y_1) \geq 1}$$

$$\text{RESOLVE} \quad \frac{x + p \geq 0 \qquad -x + q \geq 0}{p + q \geq 0}$$

Integer Satisfiability

A more expressive language

$$a_n x_n + \cdots + a_1 x_1 + a_0 \geq 0$$

What would be the equivalent of resolution?

Chvtal (1973): Cutting planes!

Integer Satisfiability

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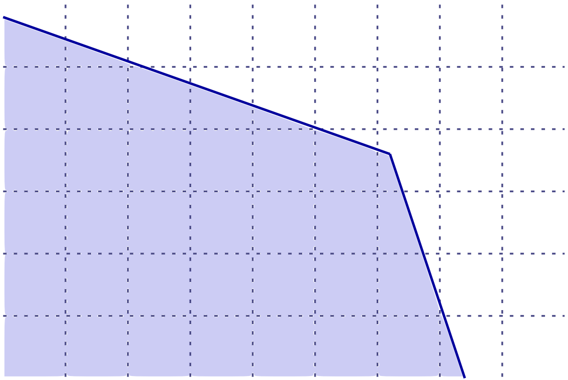
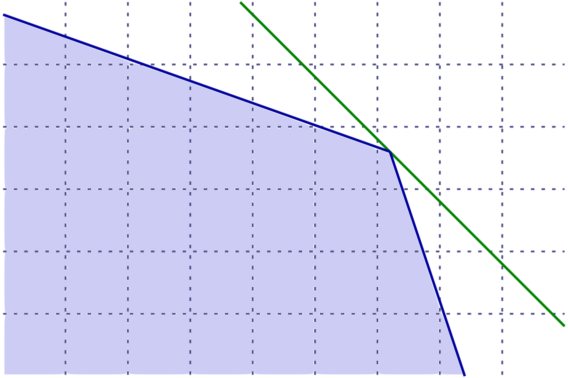
Integer Satisfiability



$$\sum_{i=1}^{n} a_i x_i + a_0 \geq 0 \qquad \sum_{i=1}^{n} b_i x_i + b_0 \geq 0$$

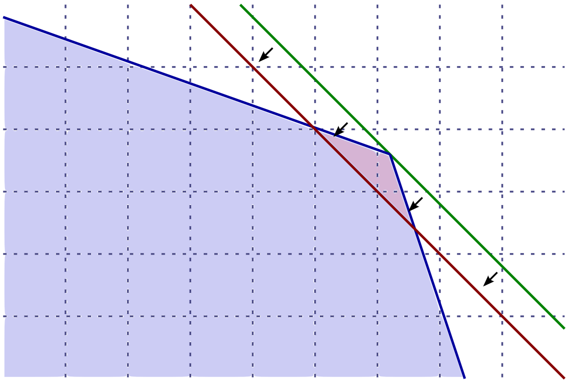
$$\sum_{i=1}^{n} (\lambda_1 a_i + \lambda_2 b_i) x_i + (\lambda_1 a_0 + \lambda_2 b_0) \geq 0$$

Integer Satisfiability



$$\frac{\sum_{i=1}^{n} a_i x_i + a_0 \geq 0 \qquad \sum_{i=1}^{n} b_i x_i + b_0 \geq 0}{\sum_{i=1}^{n}(\lambda_1 a_i + \lambda_2 b_i) x_i + (\lambda_1 a_0 + \lambda_2 b_0) \geq 0}$$

Integer Satisfiability



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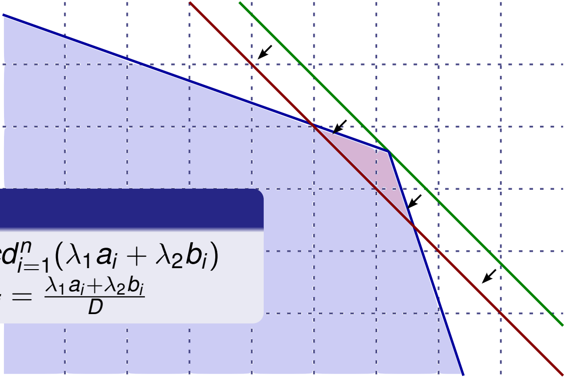
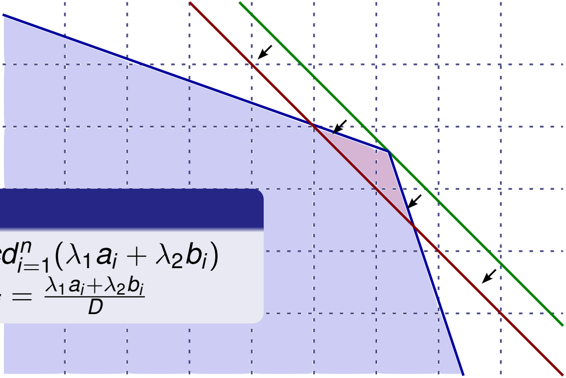


Rounding

$$D = gcd_{i=1}^{n}(\lambda_1 a_i + \lambda_2 b_i)$$
$$c_i = \frac{\lambda_1 a_i + \lambda_2 b_i}{D}$$

$$\frac{\sum_{i=1}^{n} a_i x_i + a_0 \geq 0 \qquad \sum_{i=1}^{n} b_i x_i + b_0 \geq 0}{\sum_{i=1}^{n}(\lambda_1 a_i + \lambda_2 b_i) x_i + (\lambda_1 a_0 + \lambda_2 b_0) \geq 0}$$

Integer Satisfiability



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Integer Satisfiability

General Case

We need to combine more than two planes.

$$\frac{\sum_{i=1}^{n} a_i x_i + a_0 \geq 0 \qquad \sum_{i=1}^{n} b_i x_i + b_0 \geq 0}{\sum_{i=1}^{n} c_i x_i + \lfloor c_0 \rfloor \geq 0}$$

Integer Satisfiability: CDCL?

Can we use CDCL and cutting planes to solve integers?

$$3x_3 + 2x_2 + x_1 - 4 \geq 0 \qquad x_1 = 1, x_2 = 1 \implies x_3 \geq 1$$
$$-3x_3 + x_2 + 2x_1 - 1 \geq 0 \qquad x_1 = 1, x_2 = 1 \implies x_3 \leq 0$$

$$\frac{3x_3 + 2x_2 + x_1 - 4 \geq 0 \qquad -3x_3 + x_2 + 2x_1 - 1 \geq 0}{3x_2 + 3x_1 - 5 \geq 0}$$

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\end{array}
$$

$$
\frac{3x_3 + 2x_2 + x_1 - 4 \geq 0 \qquad -3x_3 + x_2 + 2x_1 - 1 \geq 0}{3x_2 + 3x_1 - 5 \geq 0}
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$$\frac{3x_3 + 2x_2 + x_1 - 4 \geq 0 \qquad -3x_3 + x_2 + 2x_1 - 1 \geq 0}{x_2 + x_1 - 2 \geq 0}$$

Integer Satisfiability: CDCL?

Doesn't work!

If in 0-1 case, we can still eliminate the current assignment by learning $(1 - x_1) + (1 - x_2) \geq 1$.

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Integer Satisfiability: CDCL?

How come it works for clauses?

- Clauses are inequalities with unit coefficients on the resolved (propagated) variable.
- This means there is no rounding involved when propagating – inequality is **tightly-propagating**.

Outline

Desired ingredients (CDCL winners)

- Model search complemented with the generation of resolvents explaining the conflicts.
- Propagation rules enabling search space reduction and early conflict detection.
- Resolvents that enable non-chronological backtracking.
- Resolvents should be implied by the input formula, and not conditioned by any decisions.
- Resolvents should be removable, allowing for flexible memory management by keeping the constraint database limited.
- Decisions not based on a fixed variable order, enabling dynamic reordering heuristics.

Influence

- 2009 McMillan, Kuehlmann, Sagiv - Generalizing DPLL to Richer Logics
- 2009 Korovin, Tsiskaridze, Voronkov - Conflict Resolution
- 2010 Cotton - Natural Domain SMT - A Preliminary Assesment

Bound refinement sequence and Propagation

Bound sequence consists of decisions and bound refinements, with all reasons tightly-propagating.

$$C = \{\overbrace{-x + y + 1 \leq 0}^{I}, \overbrace{-y + x \leq 0}^{J}, \overbrace{-y \leq 0}^{K}\}$$

$\langle [\![y \geq_K 0]\!], C \rangle$
$\langle [\![y \geq_K 0,\ x \geq_I 1]\!], C \rangle$
$\langle [\![y \geq_K 0,\ x \geq_I 1,\ y \geq_J 1]\!], C \rangle$
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$C = \{\overbrace{-x + y + 1 \leq}^{I}$ …

Propagation loops

If some variables are unbounded, we could get an infinite loop that we have to break.

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Decisions

Decisions are bound refinements that fix a variable to one of it's bounds.

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Decisions

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Deriving Tightly-Propagating Constraints

$$C = \{\underbrace{-y \leq 0}_{I_1}, \underbrace{-x + 2 \leq 0}_{I_2}, \underbrace{-y + 7 + x \leq 0}_{I_3}, \underbrace{-3z + 2y - 5x \leq 0}_{I_4}\}$$

$$M_4 = [\![\; y \geq_{I_1} 0, \;\; x \geq_{I_2} 2, \;\; y \geq_{I_3} 9, \;\; x \leq 2 \;]\!]$$

I_4 is implying a lower bound of 3 for z!

$\langle M_4, -3z \oplus 2y - 5x \rangle$
$\langle M_3, -3z - 6x \oplus 2y + 2 \rangle$
$\langle M_2, -3z - 6x \oplus 2x + 16 \rangle$
$\langle M_1, -3z - 6x \oplus 20 \rangle$
$-z - 2x + 7 \leq 0$

Tight!

$-z - 2x + 7 \leq 0$ is tight and implies the same bound on z!

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Reasoning

$x \leq 2$ is a decided bound, M contains implied bound $x \geq_{l_2} 2$. We make the coefficient of x divisible by 3 by adding $-x + 2 \leq 0$.

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Reasoning

We eliminate y by adding two times $-y + 7 + x \le 0$.

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Reasoning

We eliminate x in $2x + 16$ by adding two times $-x + 2 \leq 0$.

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$-z - 2x + 7 \leq 0$

Reasoning

All coefficients divisible by 3, we round and finish.

Tight!

$-z - 2x + 7 \leq 0$ is tight and implies the same bound on z!

Deriving Tightly-Propagating Constraints

$$C = \{\underbrace{-y \leq 0}_{I_1}, \underbrace{-x + 2 \leq 0}_{I_2}, \underbrace{-y + 7 + x \leq 0}_{I_3}, \underbrace{-3z + 2y - 5x \leq 0}_{I_4}\}$$

$$M_4 = [\![\ y \geq_{I_1} 0,\ \ x \geq_{I_2} 2,\ \ y \geq_{I_3} 9,\ \ x \leq 2\ ]\!]$$

I_4 is implying a lower bound of 3 for z!

$\langle M_4, -3z \oplus 2y - 5x \rangle$
$\langle M_3, -3z - 6x \oplus 2y + 2 \rangle$
$\langle M_2, -3z - 6x \oplus 2x + 16 \rangle$
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Example

$$C = \{\underbrace{-x \leq 0}_{I_1},\ \underbrace{6x - 3y - 2 \leq 0}_{I_2},\ \underbrace{-6x + 3y + 1 \leq 0}_{I_3}\}$$

$$J \equiv y - 2x + 1 \leq 0$$

$\langle [\![]\!], C \rangle$

$\langle [\![x \geq_{I_1} 0]\!], C \rangle$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!], C \rangle$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!], C \rangle \vdash 6x - 3y - 2 \leq 0$

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$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!],$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!], C \rangle \vdash 6x - 3y - 2 \leq 0$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle \vdash 1 \leq 0$

Reasoning

Propagate on x using I_1.

Example

$$C = \{\underbrace{-x \leq 0}_{I_1},\ \underbrace{6x - 3y - 2 \leq 0}_{I_2},\ \underbrace{-6x + 3y + 1 \leq 0}_{I_3}\}$$

$$J \equiv y - 2x + 1 \leq 0$$

$\langle [\![ ]\!], C \rangle$

$\langle [\![x \geq_{I_1} 0]\!], C \rangle$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!], C \rangle$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!], C \rangle \vdash 6x - 3y - 2 \leq 0$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle \vdash 1 \leq 0$

Example

$$C = \{\underbrace{-x \leq 0}_{I_1},\ \underbrace{6x - 3y - 2 \leq 0}_{I_2},\ \underbrace{-6x + 3y + 1 \leq 0}_{I_3}\}$$

$$J \equiv y - 2x + 1 \leq 0$$

$\langle [\![ ]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!],$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!], C \rangle \vdash 6x - 3y - 2 \leq 0$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle \vdash 1 \leq 0$

Reasoning

Decide x.

Example

$$C = \{\underbrace{-x \leq 0}_{I_1},\ \underbrace{6x - 3y - 2 \leq 0}_{I_2},\ \underbrace{-6x + 3y + 1 \leq 0}_{I_3}\}$$

$$J \equiv y - 2x + 1 \leq 0$$

$\langle [\![]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!], C \rangle \vdash 6x - 3y - 2 \leq 0$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle \vdash 1 \leq 0$

Example

$$C = \{\underbrace{-x \leq 0}_{I_1},\ \underbrace{6x - 3y - 2 \leq 0}_{I_2},\ \underbrace{-6x + 3y + 1 \leq 0}_{I_3}\}$$

$$J \equiv y - 2x + 1 \leq 0$$

$\langle [\![ ]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!],$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!],$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle \vdash 1 \leq 0$

Reasoning

Propagate y using I_3 and obtain the tight constraint J.

Example

$$C = \{\underbrace{-x \leq 0}_{I_1},\ \underbrace{6x - 3y - 2 \leq 0}_{I_2},\ \underbrace{-6x + 3y + 1 \leq 0}_{I_3}\}$$

$$J \equiv y - 2x + 1 \leq 0$$

$\langle [\![]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!], C \rangle \vdash 6x - 3y - 2 \leq 0$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle \vdash 1 \leq 0$

Example

$$C = \{\underbrace{-x \leq 0}_{I_1},\ \underbrace{6x - 3y - 2 \leq 0}_{I_2},\ \underbrace{-6x + 3y + 1 \leq 0}_{I_3}\}$$

$$J \equiv y - 2x + 1 \leq 0$$

$\langle [\![]\!], C \rangle$

$\langle [\![x \geq_{I_1} 0]\!], C \rangle$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!],$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!], C \rangle \vdash 6x - 3y - 2 \leq 0$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle \vdash 1 \leq 0$

Reasoning

Conflict on I_2

Example

$$C = \{\underbrace{-x \leq 0}_{I_1},\ \underbrace{6x - 3y - 2 \leq 0}_{I_2},\ \underbrace{-6x + 3y + 1 \leq 0}_{I_3}\}$$

$$J \equiv y - 2x + 1 \leq 0$$

$\langle [\![]\!], C \rangle$

$\langle [\![x \geq_{I_1} 0]\!], C \rangle$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!], C \rangle$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!], C \rangle \vdash 6x - 3y - 2 \leq 0$

$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle \vdash 1 \leq 0$

Example

$$C = \{\underbrace{-x \leq 0}_{I_1},\ \underbrace{6x - 3y - 2 \leq 0}_{I_2},\ \underbrace{-6x + 3y + 1 \leq 0}_{I_3}\}$$

$$J \equiv y - 2x + 1 \leq 0$$

$\langle [\![]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!],$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!], C \rangle \vdash 6x - 3y - 2 \leq 0$
$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle \vdash 1 \leq 0$

Reasoning

We go back and resolve the conflict.

Example

$$C = \{\underbrace{-x \leq 0}_{I_1},\ \underbrace{6x - 3y - 2 \leq 0}_{I_2},\ \underbrace{-6x + 3y + 1 \leq 0}_{I_3}\}$$

$$J \equiv y - 2x + 1 \leq 0$$

$$\langle [\![ ]\!], C \rangle$$
$$\langle [\![x \geq_{I_1} 0]\!], C \rangle$$
$$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle$$
$$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!], C \rangle$$
$$\langle [\![x \geq_{I_1} 0,\ x \leq 0,\ y \leq_J -1]\!], C \rangle \vdash 6x - 3y - 2 \leq 0$$
$$\langle [\![x \geq_{I_1} 0,\ x \leq 0]\!], C \rangle \vdash 1 \leq 0$$

Termination

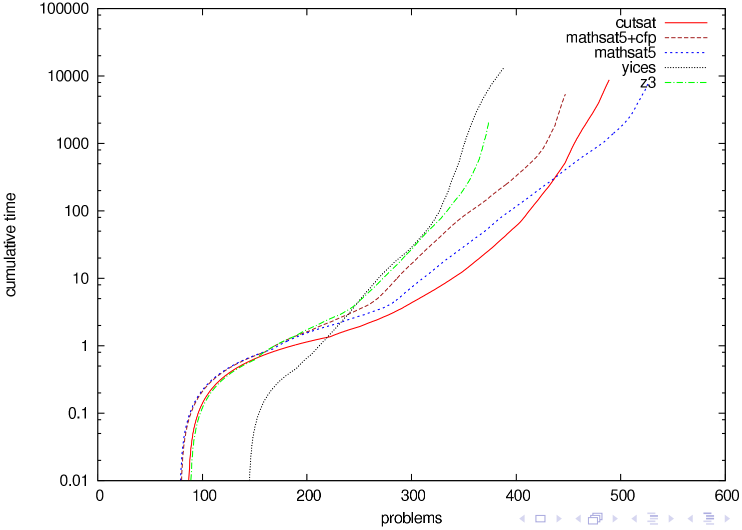
- If all the variables are bounded, termination follows easily in the manner similar to that of CDCL.
- If there are unbounded case, termination is more sofisticated.
  - Extend the procedure to divisibility constraints.
  - Introduce **strong** resolvents based on Coopers elimination procedure.
  - Block conflicts with strong resolvents and show that this can not happen indefinelty.

Outline

Experimental Results

problems	miplib2003 (16)		pb2010 (81)		dillig (250)		slacks (250)		pigeons (19)		primes (37)	
cutsat	722.78	12	1322.61	46	4012.65	223	2722.19	152	0.15	19	5.08	37
smt solvers	time(s)	solved	time(s)	solved	time(s)	solved	time(s)	solved	time(s)	solved	time(s)	solved
mathsat5+cfp	575.20	11	2295.60	33	**2357.18**	**250**	160.67	98	0.23	19	1.26	37
mathsat5	**89.49**	**11**	1224.91	38	3053.19	245	**3243.77**	**177**	0.30	19	**1.03**	**37**
yices	226.23	8	57.12	37	5707.46	159	7125.60	134	**0.07**	**19**	0.64	32
z3	532.09	9	**168.04**	**38**	885.66	171	589.30	115	0.27	19	11.19	23
pb solvers												
sat4j	**22.34**	**10**	**798.38**	**67**	0.00	0	0.00	0	110.81	8	0.00	0
sat4j+cp	28.56	10	349.15	60	0.00	0	0.00	0	**4.85**	**19**	0.00	0
mip solvers												
glpk	242.67	12	1866.52	46	4.50	248	0.08	10	**0.09**	**19**	**0.44**	**37**
cplex	53.86	15	1512.36	58	8.65	250	8.76	248	0.51	19	3.47	37
gurobi	**28.96**	**15**	**1332.53**	**58**	**5.48**	**250**	**8.12**	**248**	0.21	19	0.80	37

Experimental Results



Ingredients achieved!

- Model search with conflict analysis
- Propagation enabling search space reduction
- Non-chronological backtracking
- Globally valid resolvents
- Resolvents can be removed be removable
- Dynamic reordering heuristics

Conclusion

- A new SAT-like procedure for linear integer problems.
- Novel *dynamic* cut procedure for deriving tight constraints.
- Performs competitively with the SMT solvers.
- Will be available at `http://www.cs.nyu.edu/~dejan/cutsat/`.