Description Logics

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1. Motivation and introduction to Description Logics
2. Tableau-based reasoning procedures
3. Automata-based reasoning procedures
4. Complexity of reasoning in Description Logics
5. Reasoning in inexpressive Description Logics

Reasoning procedures

1. The procedure should be a decision procedure for the problem.
2. The procedure should be as efficient as possible:
   preferably optimal w.r.t. the (worst-case) complexity of the problem
3. The procedure should be practical:
   easy to implement and optimize, and behave well in applications

The tableau-based resoning procedure for $\mathcal{ALC}$

- satisfies the first requirement, as shown in the previous lecture.
- Highly-optimized implementations in systems like FaCT and RACER demonstrate that it satisfies the third requirement.
- It does not satisfy the second requirement in the presence of GCIs.

Tableau-based procedures

disadvantages

- the consistency problem for $\mathcal{ALC}$ with GCIs is ExpTime-complete, but it is very hard to design a tableau-based algorithm that is better than NExpTime:
  - exponentially long chains of role successors may be generated before blocking occurs
  - to each individual in the chain, non-deterministic rules may be applied
- termination requires blocking:
  - proof of termination and soundness becomes more complicated
  - for more expressive DLs (e.g., with number restrictions and inverse roles) one needs sophisticated blocking conditions

Automata-based procedures

general idea

For simplicity, we restrict the attention to satisfiability,
i.e., consistency of an ABox of the form $\{C_0(a_0)\}$ w.r.t. a general TBox $\mathcal{T}$.

- Show that C_0 is satisfiable w.r.t. $\mathcal{T}$ iff $\mathcal{T}$ and $\{C_0(a_0)\}$ have a tree-shaped model with root a_0.
- Translate $C_0, \mathcal{T}$ into a tree automaton $\mathcal{A}_{C_0,\mathcal{T}}$ that accepts exactly the tree-shaped models of $\mathcal{T}$ and $\{C_0(a_0)\}$
- Test $\mathcal{A}_{C_0,\mathcal{T}}$ for emptiness: is there a tree accepted by $\mathcal{A}_{C_0,\mathcal{T}}$?

Automata-based procedures

advantages and disadvantages

+ separation between DL-dependent part (translation) from DL-independent part (emptiness test)
+ termination is not an issue if we use automata working on infinite trees
+ well-suited for showing ExpTime upper-bounds: translation is exponential, emptiness test polynomial

− usually also best-case exponential: translation required before emptiness test can be applied
− no optimized implementations available

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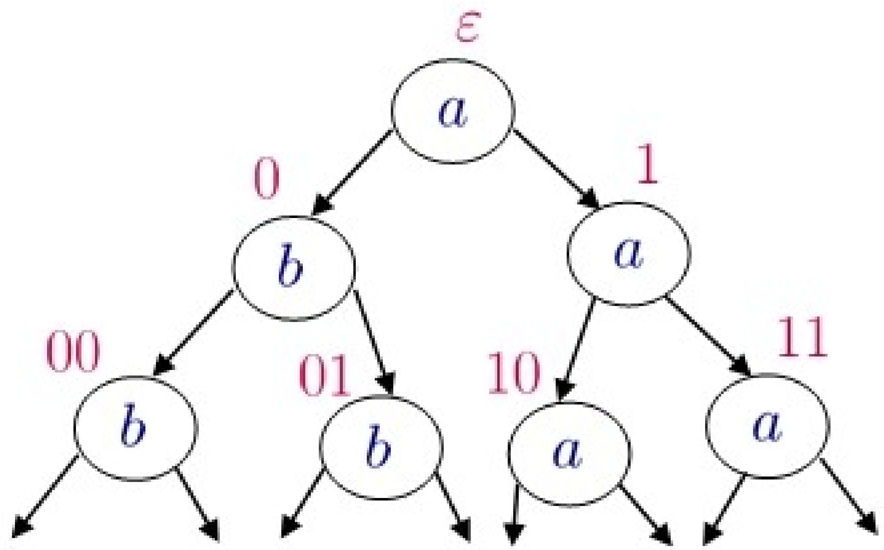
Infinite trees

definition

We consider infinite trees with a fixed out-degree k, whose nodes are labeled with elements from a finite alphabet Σ:

Example: $k = 2$ and $\Sigma = \{a, b\}$



this tree is described by the mapping $t : \{0,1\}^* \to \Sigma$ with

$$t(u) := \begin{cases} b & \text{if } u \text{ starts with } 0 \\ a & \text{otherwise} \end{cases}$$

k-ary tree over Σ:

$t : \{0, \ldots, k-1\}^* \to \Sigma$

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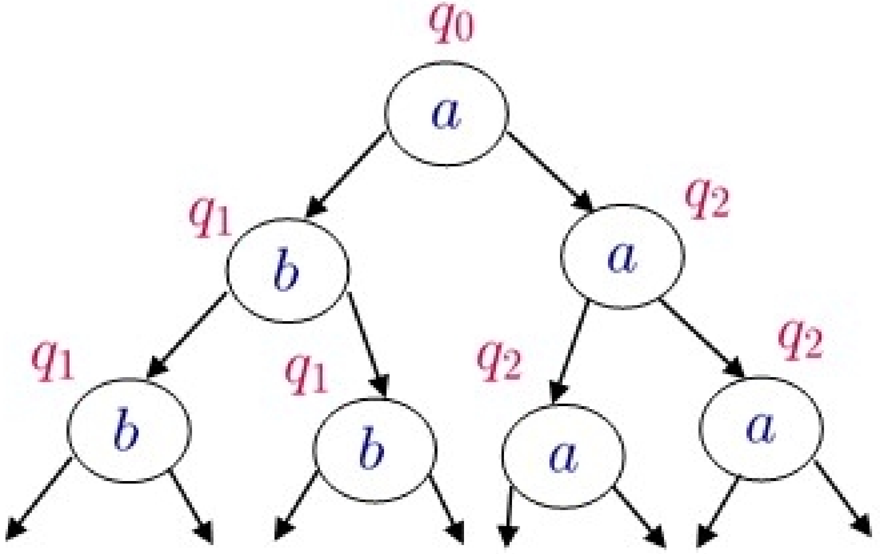
Automata on infinite trees

informal description

The automaton labels nodes of the tree with states.



$Q = \{q_0, q_1, q_2\}$

$I = \{q_0\}$

$(q_0, a) \to (q_1, q_2) \quad (q_0, a) \to (q_2, q_1)$

$(q_1, b) \to (q_1, q_1)$

$(q_2, a) \to (q_2, q_2)$

The root is labeled with an initial state.

The labeling of the other nodes must be compatible with the transition relation.

The transition relation may be non-deterministic.

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Automata on infinite trees

formal description

A looping automaton working on k-ary trees is of the form $\mathcal{A} = (Q, \Sigma, I, \Delta)$ where

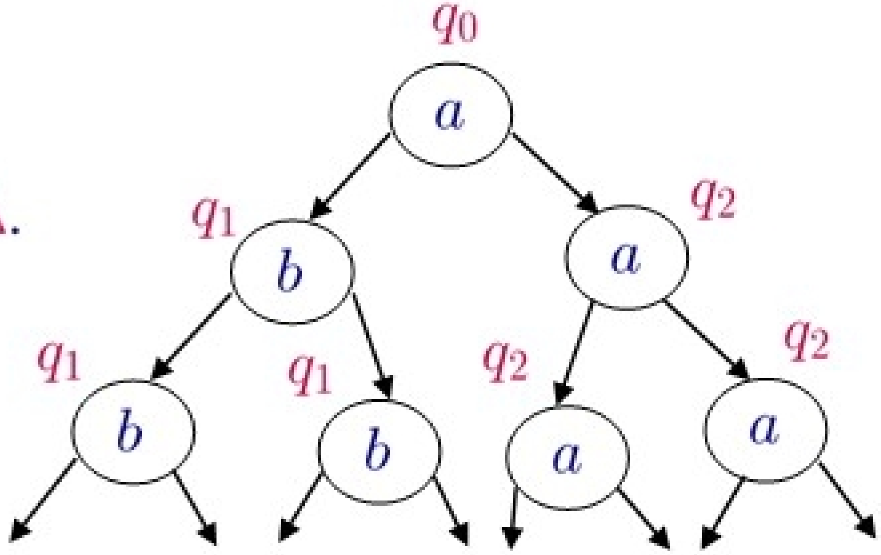
- Q is a finite set of states, and $I \subseteq Q$ the set of initial states;
- Σ is a finite alphabet;
- $\Delta \subseteq Q \times \Sigma \times Q^k$ is the transition relation.

A run of this automaton on a k-ary tree
$t : \{0, \ldots, k-1\}^* \to \Sigma$ is a k-ary tree
$r : \{0, \ldots, k-1\}^* \to Q$ such that

- $(r(u), t(u)) \to (r(u0), \ldots, r(u(k-1))) \in \Delta$.

The run is called initial if

- $r(\varepsilon) \in I$.



Looping automaton: no additional condition based on accepting states

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Accepted tree language

The tree language accepted by the looping automaton $\mathcal{A}$ is

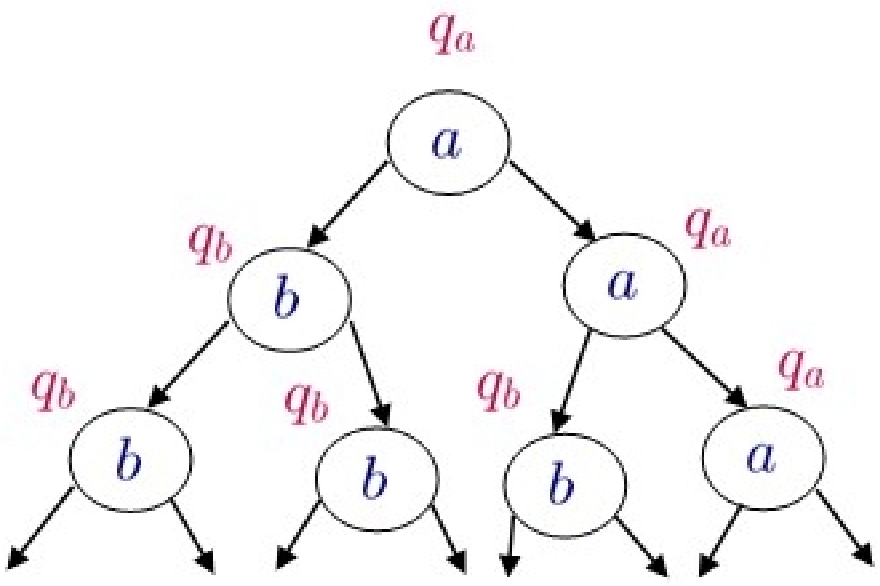
$$L(\mathcal{A}) := \{t \mid \text{there is an initial run of } \mathcal{A} \text{ on the } k\text{-ary tree } t\}$$

Consider the following binary tree language over $\Sigma = \{a, b\}$:

$$L := \{t \mid a \text{ never occurs below a } b \text{ in } t\}$$

$\mathcal{A} = (Q, \Sigma, I, \Delta)$ with

- $Q := \{q_a, q_b\}$;
- $I := \{q_a, q_b\}$;
- $\Delta := \{(q_b, b) \rightarrow (q_b, q_b)\} \cup \{(q_a, a) \rightarrow (q, q') \mid q, q' \in Q\}$



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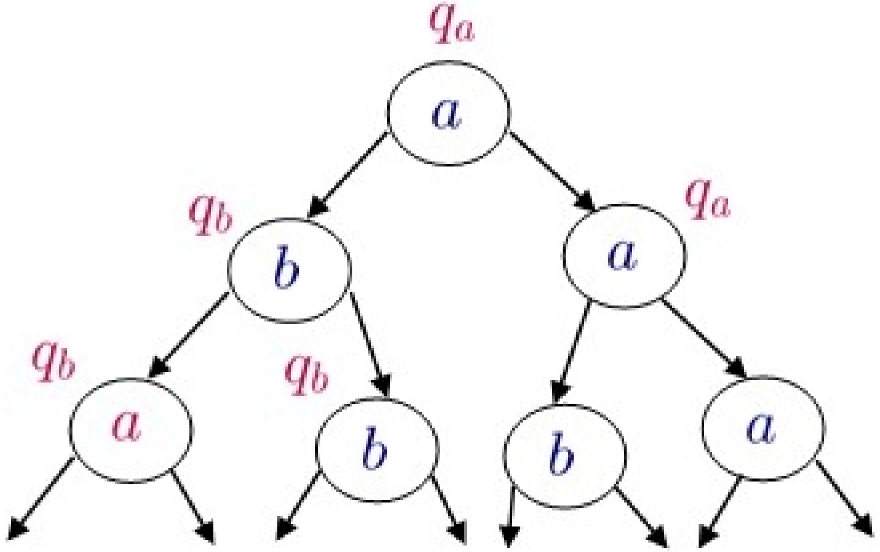

The emptiness problem

for looping tree automata

Given: a looping tree automaton $\mathcal{A}$

Question: is $L(\mathcal{A}) = \emptyset$?

Top-down approach:

- label root with an initial state;
- apply transition relation to label successor nodes.

Problem:

- termination requires blocking if states are repeated on a path;
- if the automaton is non-deterministic, then we must consider all possibile initial states and transitions.

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The emptiness test

Bottom-up approach

- Compute all bad states, i.e., states that cannot occur in a run.
- $L(\mathcal{A}) = \emptyset$ iff all initial states are bad.

$\text{Bad}_0(\mathcal{A}) := \emptyset$

$\text{Bad}_1(\mathcal{A}) := \{q \mid \text{there is no transition } (q, \cdot) \rightarrow (\cdots)\}$

$i := 1$

while $\text{Bad}_i(\mathcal{A}) \neq \text{Bad}_{i-1}(\mathcal{A})$ do

$$\text{Bad}_{i+1}(\mathcal{A}) := \text{Bad}_i(\mathcal{A}) \cup \{q \mid \text{for all transitions } (q, \cdot) \rightarrow (q_1, \ldots, q_k) \text{ there is } j \text{ with } q_j \in \text{Bad}_i(\mathcal{A})\}$$

$i := i + 1$

od

Answer "empty" iff $I \subseteq \text{Bad}_i(\mathcal{A})$

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The emptiness test

Bottom-up approach

The algorithm decides the emptiness problem in polynomial time:

- the while-loop always terminates after at most $|Q|$ iterations:

  $$\mathrm{Bad}_0(\mathcal{A}) \subseteq \mathrm{Bad}_1(\mathcal{A}) \subseteq \mathrm{Bad}_2(\mathcal{A}) \subseteq \ldots \subseteq \mathrm{Bad}_k(\mathcal{A}) = \mathrm{Bad}_{k+1}(\mathcal{A})$$

  for some $k \leq |Q|$;

- every single iteration of the loop can be done in polynomial time;

- if $q \in \mathrm{Bad}_i(\mathcal{A})$ for some $i \geq 0$ then q cannot occur in a run of $\mathcal{A}$;

- if $q \notin \mathrm{Bad}_k(\mathcal{A})$ then there is a run containing q as label of the root;
  for some tree

- if $i \in I \setminus \mathrm{Bad}_k(\mathcal{A})$ then there is an initial run.

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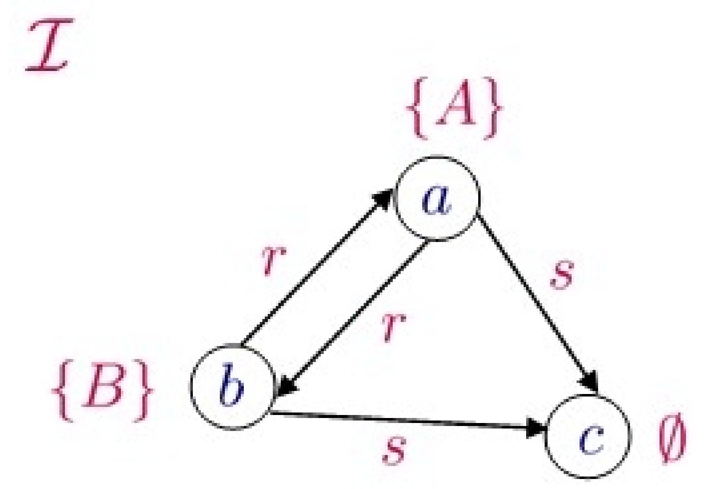
Tree model property of $\mathcal{ALC}$.

Interpretations can be viewed as graphs:

- nodes are the elements of $\Delta^{\mathcal{I}}$;
- interpretation of roles yields edges;
- interpretation of concepts yields node labels.



model of

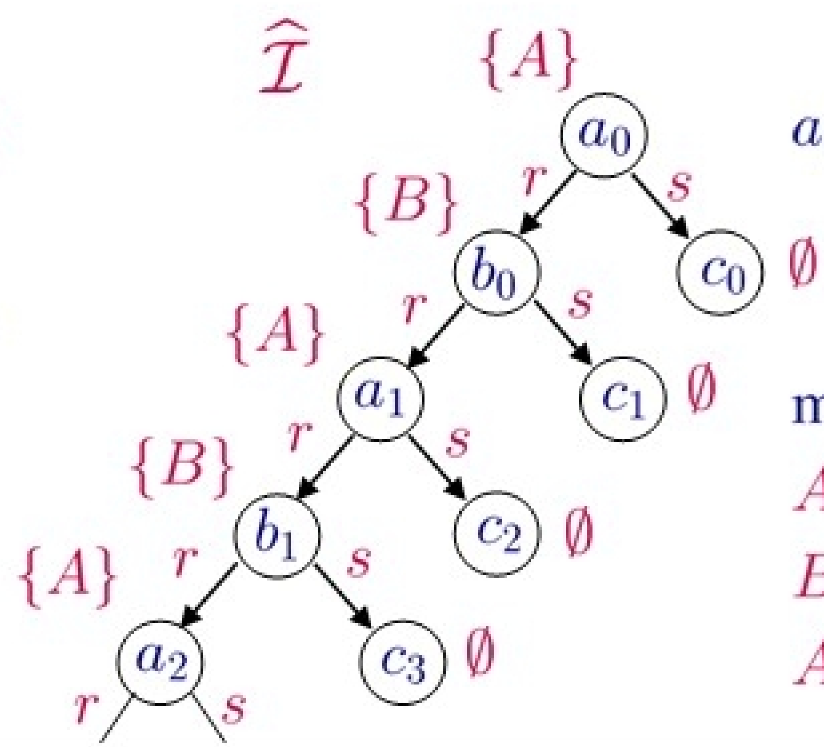
$A \sqsubseteq \exists r.B$

$B \sqsubseteq \exists r.A$

$A \sqcup B \sqsubseteq \exists s.\top$

$a \in A^{\mathcal{I}}$

Starting with a given node, the graph can be unraveled into a tree without "changing membership" in concepts.



$a_0 \in A^{\widehat{\mathcal{I}}}$

model of

$A \sqsubseteq \exists r.B$

$B \sqsubseteq \exists r.A$

$A \sqcup B \sqsubseteq \exists s.\top$

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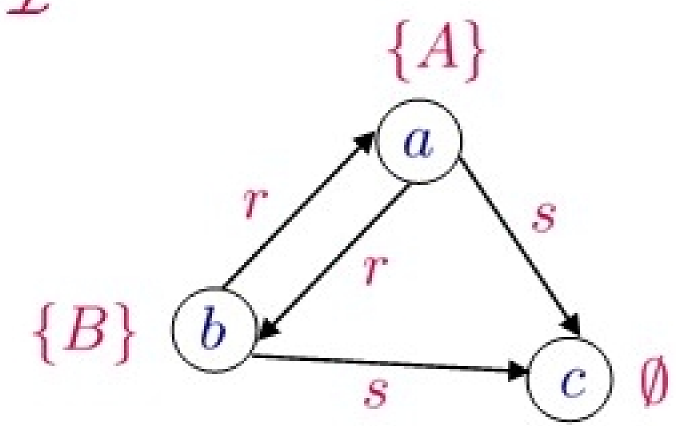
Tree model property of $\mathcal{ALC}$.

$\mathcal{T}$ general $\mathcal{ALC}$-TBox, C $\mathcal{ALC}$-concept:

> C is satisfiable w.r.t. $\mathcal{T}$
>
> iff
>
> there is a tree model of $\mathcal{T}$ whose root belongs to C

$\mathcal{I}$



model of

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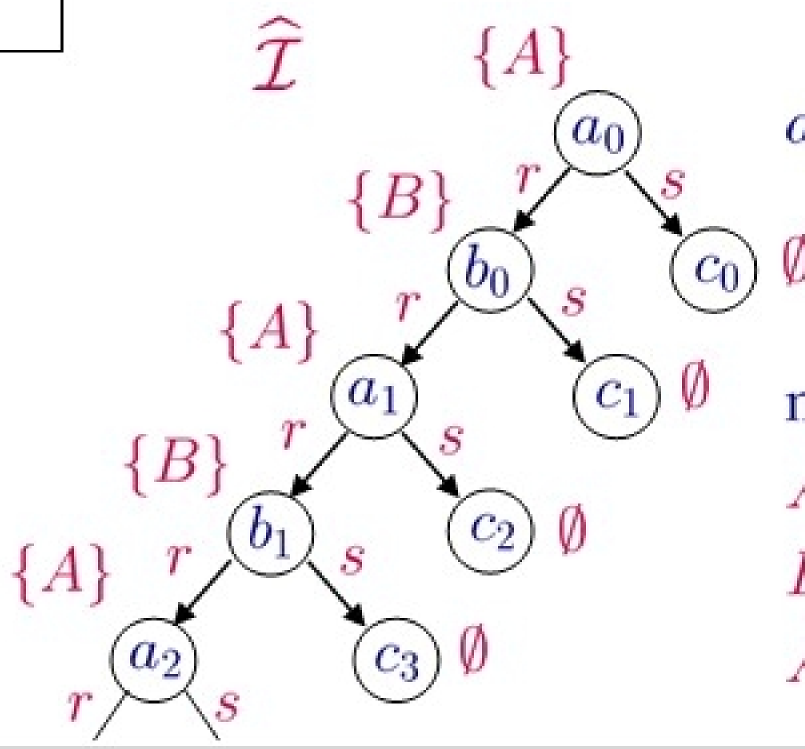

$a_0 \in A^{\widehat{\mathcal{I}}}$

model of

$A \sqsubseteq \exists r.B$

$B \sqsubseteq \exists r.A$

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Subdescriptions

of $\mathcal{ALC}$-concept descriptions

- $C \in N_C$: $\text{Sub}(A) := \{A\}$ for $A \in N_C$;
- $C = C_1 \sqcap C_2$ or $C = C_1 \sqcup C_2$: $\text{Sub}(C) := \{C\} \cup \text{Sub}(C_1) \cup \text{Sub}(C_2)$;
- $C = \neg D$ or $C = \exists r.D$ or $C = \forall r.D$: $\text{Sub}(C) := \{C\} \cup \text{Sub}(D)$.

$$\text{Sub}(A \sqcap \exists r.(A \sqcup B)) = \{A \sqcap \exists r.(A \sqcup B),\ A,\ \exists r.(A \sqcup B),\ A \sqcup B,\ B\}$$

$$\text{Sub}(\mathcal{T}) := \bigcup_{C \sqsubseteq D \in \mathcal{T}} \text{Sub}(C) \cup \text{Sub}(D)$$

- the cardinality of $\text{Sub}(C)$ is bounded by the size of C;
- the size of the elements of $\text{Sub}(C)$ is bounded by the size of C;
- cardinality and size of $\text{Sub}(\mathcal{T})$ is polynomial in the size of $\mathcal{T}$.

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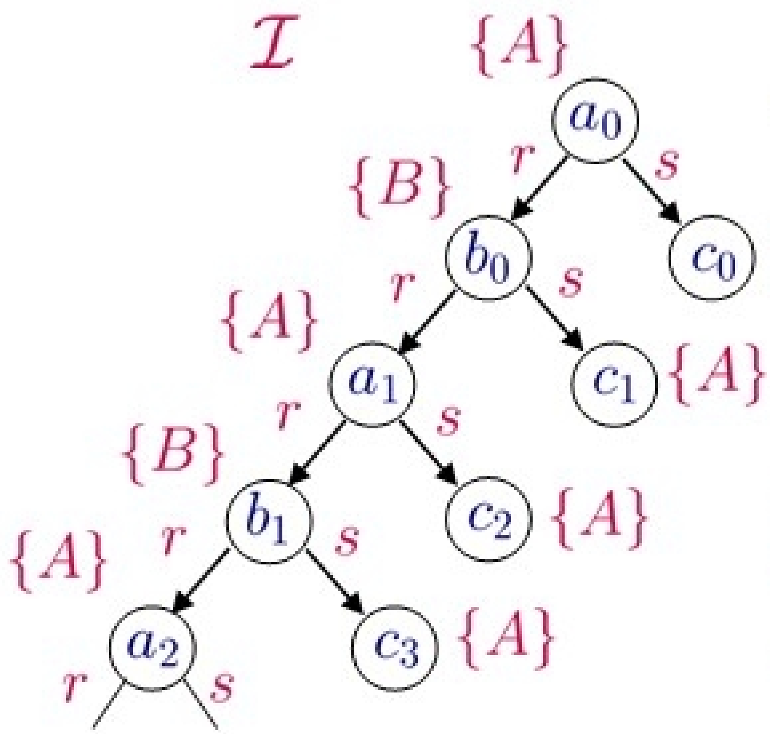
Extension of tree models

to trees labeled with subdescriptions

Let $\mathcal{T}$ be a general TBox, C_0 a concept description, and $\mathcal{I}$ a tree model of $\mathcal{T}$ whose root belongs to C_0.

Extend node labels to subdescriptions from $S := \mathsf{Sub}(\mathcal{T}) \cup \mathsf{Sub}(C_0)$:

$$\ell(d) := \{C \in S \mid d \in C^{\mathcal{I}}\}.$$



$a_0 \in A^{\mathcal{I}}$

model of
$A \sqsubseteq \exists r.B$
$B \sqsubseteq \exists r.A$
$A \sqcup B \sqsubseteq \exists s.A$

$\ell(a_0) = \{A, A \sqcup B, \exists r.B, \exists s.A\}$

$\ell(b_0) = \{B, A \sqcup B, \exists r.A, \exists s.A\}$

$\ell(c_0) = \{A, A \sqcup B\}$

$\mathsf{Sub}(\mathcal{T}) \cup \mathsf{Sub}(A) = \{A, \exists r.B, B, \exists r.A, A \sqcup B, \exists s.A\}$

Tree automaton

main idea

Given $\mathcal{T}$ and C_0, construct a looping automaton that accepts the extended tree models of $\mathcal{T}$ whose root label contains C_0.

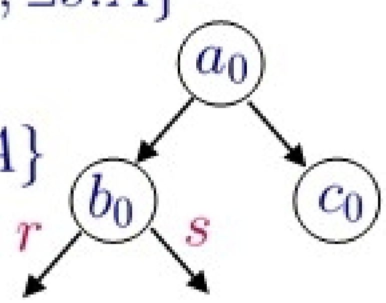
<u>Problem:</u> mismatch between the underlying kinds of trees

1. Edge labels: extended tree models have roles as edge labels, automata work on trees without edge labels

   Solution: add role names to node label of successors

$\{r, A, A \sqcup B, \exists r.B, \exists s.A\}$

$\{r, B, A \sqcup B, \exists r.A, \exists s.A\}$



$\{s, A, A \sqcup B\}$

Tree automaton

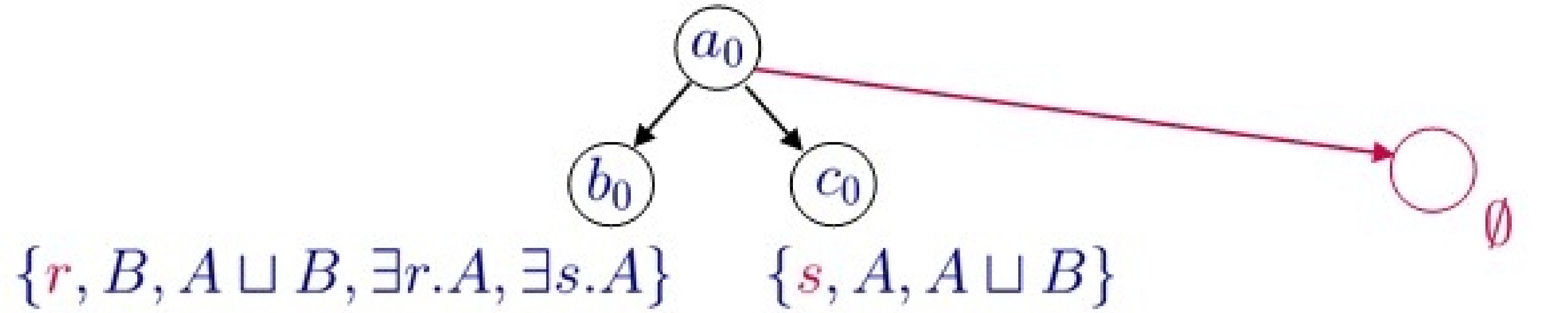
Problem: mismatch between the underlying kinds of trees

2. Varying arity: extended tree models have no fixed number of successors, automata work on trees with fixed arity k

   Solution: take as k the number of all existential restrictions in S

   $$S = \{A, \exists r.B, B, \exists r.A, A \sqcup B, \exists s.A\} \longrightarrow k = 3$$

   - a given tree model can be modified into one where nodes have at most k successors
   - for missing successors we can generated dummies



Preliminaries

required to define the trees that our automata are supposed to accept

Let $\mathcal{T}$ be a general TBox and C_0 a concept description.

Normalization 1:

Without loss of generality we assume that the GCIs in $\mathcal{T}$ are of the form $\top \sqsubseteq D$:

$$C \sqsubseteq D \text{ can be replaced by } \top \sqsubseteq \neg C \sqcup D$$

Normalization 2:

Without loss of generality we assume that C_0 and all concept descriptions in $\mathcal{T}$ are in negation normal form (NNF).

We define

$$S := \text{Sub}(\mathcal{T}) \cup \text{Sub}(C_0)$$
$$k := \text{card}(\{C \in S \mid C \text{ is an existential restriction}\})$$

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Hintikka trees

the trees that our automata are supposed to accept

The node labels of these trees are Hintikka sets.

A set $L \subseteq S \cup N_R$ is called Hintikka set if $L = \emptyset$ or

- L contains exactly one role name occurring in S;
- if $\top \sqsubseteq D \in \mathcal{T}$ then $D \in L$;
- if $C \sqcap D \in L$ then $\{C, D\} \subseteq L$;
- if $C \sqcup D \in L$ then $\{C, D\} \cap L \neq \emptyset$;
- $\{A, \neg A\} \not\subseteq L$ for all concept names A.

$\mathcal{H}$

set of all Hintikka sets

Hintikka trees

the trees that our automata are supposed to accept

The k-ary tree $h : \{0, \ldots, k-1\}^* \to \mathcal{H}$ is a Hintikka tree for $\mathcal{T}$ and C_0 if

- $C_0 \in h(\varepsilon)$;
- For all nodes u, the tuple $(h(u), h(u0), \ldots, h(u(k-1)))$ satisfies the following Hintikka successor conditions:
  - if $h(u) = \emptyset$ then $h(ui) = \emptyset$ for all $i \in \{0, \ldots, k-1\}$;
  - if $\exists r.C \in h(u)$ then there is an i with $\{C, r\} \subseteq h(ui)$;
  - if $\forall r.C \in h(u)$ and $r \in h(ui)$ then $C \in h(ui)$.

C_0 is satisfiable w.r.t. $\mathcal{T}$

iff

there is a Hintikka tree for $\mathcal{T}$ and C_0

Tree automaton

accepting the Hintikka trees for $\mathcal{T}$ and C_0

$\mathcal{A}_{C_0,\mathcal{T}} := (Q, \Sigma, I, \Delta)$ where

- $Q := \Sigma := \mathcal{H}$; states and node labels are Hintikka sets
- $I := \{L \in Q \mid C_0 \in L\}$; initial states contain C_0
- $\Delta := \{(q, \sigma, q_0, \ldots, q_{k-1}) \in Q \times \Sigma \times Q^k \mid$

  $\underline{q = \sigma}$ and $(q, q_0, \ldots, q_{k-1})$ satisfies the Hintikka successor condition$\}$

  run identical to tree

The k-ary tree $h : \{0, \ldots, k-1\}^* \to \mathcal{H}$ is accepted by $\mathcal{A}_{C_0,\mathcal{T}}$

iff

it is a Hintikka tree for $\mathcal{T}$ and C_0

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Main result

> Satisfiability of $\mathcal{ALC}$-concept descriptions w.r.t. general $\mathcal{ALC}$-TBoxes can be decided in exponential time.

1. C_0 is satisfiable w.r.t. $\mathcal{T}$ iff there is a Hintikka tree for $\mathcal{T}$ and C_0
   iff $L(\mathcal{A}_{C_0,\mathcal{T}}) \neq \emptyset$
2. The size of $\mathcal{A}_{C_0,\mathcal{T}}$ is exponential in the size of C_0 and $\mathcal{T}$.
3. The emptiness test is polynomial in the size of $\mathcal{A}_{C_0,\mathcal{T}}$.

Note:

> this bound is worst-case optimal since one can show ExpTime hardness of the problem