Verifying Continuous-Time Markov Chains
Lecture 3+4: Continuous-Time Markov Chains

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Overview

1. Negative exponential distributions
2. What are continuous-time Markov chains?
3. Transient distribution
4. Timed reachability probabilities
5. Verifying continuous stochastic CTL
6. Verifying linear real-time properties

Time in discrete-time Markov chains

The advance of time in DTMCs

- Time in a DTMC proceeds in discrete steps
- Two possible interpretations:
  1. accurate model of (discrete) time units
     - e.g., clock ticks in model of an embedded device
  2. time-abstract
     - no information assumed about the time transitions take
- State residence time is geometrically distributed

Continuous-time Markov chains

- dense model of time
- transitions can occur at any (real-valued) time instant
- state residence time is (negative) exponentially distributed
Continuous random variables

- \( X \) is a random variable (r.v., for short)
  - on a sample space with probability measure \( Pr \)
  - assume the set of possible values that \( X \) may take is dense
- \( X \) is continuously distributed if there exists a function \( f(x) \) such that:
  \[
  F_X(d) = Pr\{X \leq d\} = \int_{-\infty}^{d} f(x) \, dx \quad \text{for each real number } d
  \]
  where \( f \) satisfies: \( f(x) \geq 0 \) for all \( x \) and \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \)
- \( F_X(d) \) is the (cumulative) probability distribution function
- \( f(x) \) is the probability density function

Exponential pdf and cdf

The higher \( \lambda \), the faster the cdf approaches 1.

Negative exponential distribution

Density of exponential distribution

The density of an exponentially distributed r.v. \( Y \) with rate \( \lambda \in \mathbb{R}_{>0} \) is:

\[
f_Y(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0 \quad \text{and } f_Y(x) = 0 \text{ otherwise}
\]

The cumulative distribution of r.v. \( Y \) with rate \( \lambda \in \mathbb{R}_{>0} \) is:

\[
F_Y(d) = \int_0^d \lambda e^{-\lambda x} \, dx = \left[-e^{-\lambda x}\right]_0^d = 1 - e^{-\lambda d}.
\]

The rate \( \lambda \in \mathbb{R}_{>0} \) uniquely determines an exponential distribution.

Variance and expectation

Let r.v. \( Y \) be exponentially distributed with rate \( \lambda \in \mathbb{R}_{>0} \). Then:

- Expectation \( E[Y] = \int_0^\infty x \cdot \lambda e^{-\lambda x} \, dx = \frac{1}{\lambda} \)
- Variance \( Var[Y] = \int_0^\infty (x - E[X])^2 \lambda e^{-\lambda x} \, dx = \frac{1}{\lambda^2} \)

Why exponential distributions?

- Are adequate for many real-life phenomena
  - the time until a radioactive particle decays
  - the time between successive car accidents
  - inter-arrival times of jobs, telephone calls in a fixed interval
- Are the continuous counterpart of the geometric distribution
- Heavily used in physics, performance, and reliability analysis
- Can approximate general distributions arbitrarily closely
- Yield a maximal entropy if only the mean is known
Memoryless property

**Theorem**

1. For any exponentially distributed random variable $X$:
   
   \[ \Pr\{X > t + d \mid X > t\} = \Pr\{X > d\} \text{ for any } t, d \in \mathbb{R}_{\geq 0}. \]

2. Any cdf which is memoryless is a negative exponential one.

**Proof**

Proof of 1.: Let $\lambda$ be the rate of $X$’s distribution. Then we derive:

\[
\Pr\{X > t + d \mid X > t\} = \frac{\Pr\{X > t + d \cap X > t\}}{\Pr\{X > t\}} = \frac{\Pr\{X > t + d\}}{\Pr\{X > t\}} \nonumber \\
= \frac{e^{-\lambda(t+d)}}{e^{-\lambda t}} = e^{-\lambda d} = \Pr\{X > d\}.
\]

Proof of 2.: By contraposition, using the total law of probability.

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Closure under minimum

**Minimum closure theorem**

For independent, exponentially distributed random variables $X$ and $Y$ with rates $\lambda, \mu \in \mathbb{R}_{>0}$, the r.v. $\min(X, Y)$ is exponentially distributed with rate $\lambda+\mu$, i.e.,:

\[ \Pr\{\min(X, Y) \leq t\} = 1 - e^{-(\lambda+\mu)t} \text{ for all } t \in \mathbb{R}_{\geq 0}. \]

**Proof**

Let $\lambda$ ($\mu$) be the rate of $X$’s ($Y$’s) distribution. Then we derive:

\[
\Pr\{\min(X, Y) \leq t\} = \Pr_{X,Y}\{(x,y) \in \mathbb{R}^2_{\geq 0} \mid \min(x, y) \leq t\} \\
= \int_0^\infty \left( \int_0^\infty I_{\min(x,y)\leq t}(x,y) \cdot \lambda e^{-\lambda x} \cdot \mu e^{-\mu y} \, dy \right) \, dx \\
= \int_0^t \int_x^\infty \lambda e^{-\lambda x} \cdot \mu e^{-\mu y} \, dy \, dx + \int_0^t \int_y^\infty \lambda e^{-\lambda x} \cdot \mu e^{-\mu y} \, dx \, dy \\
= \int_0^t \lambda e^{-\lambda x} \cdot e^{-\mu x} \, dx + \int_0^t e^{-\lambda y} \cdot \mu e^{-\mu y} \, dy \\
= \int_0^t \lambda e^{-(\lambda+\mu)x} \, dx + \int_0^t \mu e^{-(\lambda+\mu)y} \, dy \\
= \int_0^t (\lambda+\mu) \cdot e^{-(\lambda+\mu)z} \, dz = 1 - e^{-(\lambda+\mu)t}
\]

---

**Generalization of the proof for the case of two exponential distributions.**
Winning the race with two competitors

The minimum of two exponential distributions

For independent, exponentially distributed random variables $X$ and $Y$ with rates $\lambda, \mu \in \mathbb{R}_{>0}$, it holds:

\[
Pr\{X \leq Y\} = \frac{\lambda}{\lambda + \mu}.
\]

Proof:

Let $\lambda (\mu)$ be the rate of $X$'s ($Y$'s) distribution. Then we derive:

\[
Pr\{X \leq Y\} = Pr_{X,Y}\{(x,y) \in \mathbb{R}^2_+ \mid x \leq y\}
\]

\[
= \int_0^\infty \mu e^{-\mu y} \left( \int_0^y \lambda e^{-\lambda x} \, dx \right) \, dy
\]

\[
= \int_0^\infty \mu e^{-\mu y} \left( 1 - e^{-\lambda y} \right) \, dy
\]

\[
= 1 - \int_0^\infty \mu e^{-\mu y} e^{-\lambda y} \, dy = 1 - \int_0^\infty \mu e^{-(\mu+\lambda)y} \, dy
\]

\[
= 1 - \frac{\mu}{\mu + \lambda} \cdot \int_0^\infty (\mu+\lambda) e^{-(\mu+\lambda) y} \, dy
\]

\[
= 1 - \frac{\mu}{\mu + \lambda} = \frac{\lambda}{\mu + \lambda}.
\]

Winning the race with many competitors

The minimum of several exponentially distributed r.v.'s

For independent, exponentially distributed random variables $X_1, X_2, \ldots, X_n$ with rates $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}_{>0}$ it holds:

\[
Pr\{X_i = \min(X_1, \ldots, X_n)\} = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}.
\]

Proof:

Generalization of the proof for the case of two exponential distributions.

Overview

- Negative exponential distributions
- What are continuous-time Markov chains?
- Transient distribution
- Timed reachability probabilities
- Verifying continuous stochastic CTL
- Verifying linear real-time properties
Continuous-time Markov chain

A CTMC is a tuple \((S, P, r, \tau_{init}, AP, L)\) where
- \((S, P, \tau_{init}, AP, L)\) is a DTMC, and
- \(r : S \rightarrow \mathbb{R}_{>0}\), the exit-rate function

**Interpretation**
- residence time in state \(s\) is exponentially distributed with rate \(r(s)\).
- phrased alternatively, the average residence time of state \(s\) is \(\frac{1}{r(s)}\).
- thus, the higher the rate \(r(s)\), the shorter the average residence time in \(s\).

Example: a classical perspective

\[
\begin{align*}
\text{r}(s) &= 25, \quad \text{r}(t) = 4, \quad \text{r}(u) = 2 \quad \text{and} \quad \text{r}(v) = 100
\end{align*}
\]

The transition rate \(R(s, s') = P(s, s') \cdot r(s)\)

We use \((S, P, r, \tau_{init}, AP, L)\) and \((S, R, \tau_{init}, AP, L)\) interchangeably.

CTMC semantics by example

**CTMC semantics**
- Transition \(s \rightarrow s' := r.v. \ X_{s,s'}\) with rate \(R(s, s')\)
- Probability to go from state \(s_0\) to, say, state \(s_2\) is:
  \[
  \Pr\{X_{s_0, s_2} \leq X_{s_0, s_1} \cap X_{s_0, s_2} \leq X_{s_0, s_3}\} = \frac{R(s_0, s_2)}{R(s_0, s_1) + R(s_0, s_2) + R(s_0, s_3)} = \frac{R(s_0, s_2)}{r(s_0)}
  \]
- Probability of staying at most \(t\) time in \(s_0\) is:
  \[
  \Pr\{\min(X_{s_0, s_2}, X_{s_0, s_1}, X_{s_0, s_3}) \leq t\} = 1 - e^{-\left(R(s_0, s_2) + R(s_0, s_2) + R(s_0, s_3)\right) t} = 1 - e^{-r(s_0) t}
  \]
CTMC semantics

Enabledness
The probability that transition \( s \to s' \) is enabled in \([0, t]\) is \( 1 - e^{-R(s,s') \cdot t} \).

State-to-state timed transition probability
The probability to move from non-absorbing \( s \) to \( s' \) in \([0, t]\) is:
\[
\frac{R(s, s')}{r(s)} \cdot (1 - e^{-r(s) \cdot t}).
\]

Residence time distribution
The probability to take some outgoing transition from \( s \) in \([0, t]\) is:
\[
\int_0^t r(s) e^{-r(s) \cdot x} \, dx = 1 - e^{-r(s) \cdot t}.
\]

Enzyme-catalysed substrate conversion

Kinetics

<table>
<thead>
<tr>
<th>Reaction</th>
<th>Reaction mechanism</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E + S \to ES \to E + P )</td>
<td>Enzyme-catalysed reaction</td>
</tr>
</tbody>
</table>

Enzyme-catalysis is the investigation of how enzymes turn substrates and turn them into products. The rate data used in kinetic analysis are commonly obtained from enzyme assays, where since the 1960s, the dynamics of many enzymes are studied on the level of individual molecules.

In 1902 Victor Henri proposed a quantitative theory of enzyme kinetics, but his experimental data were not useful because the significance of the hydrogen ion concentration was not yet appreciated. After Paul Léonard Saramaze had defined the logarithmic (pH)-rate and introduced the concept of buffering in 1903, the German chemists Leonor Michaelis and his Canadian assistant Mortimer Mosenthal repeated Henri's experiments and confirmed his equation which is referred to as Henri-Michaelis-Menten kinetics (or Michaelis-Menten kinetics). Their work was further developed by G. E. Briggs and J. R. H. Haldane, who derived kinetic equations that are still widely considered today a starting point in solving enzymic activity.

The major contribution of Henri was to think of enzyme reactions in two stages. In the first, the substrate binds reversibly to the enzyme, forming the enzyme-substrate complex. This is sometimes called the Michaelis complex. The enzyme then catalyzes the chemical step in the reaction and releases the product. Note that the simple Michaelis-Menten mechanism for the enzymatic activity is considered today a basic idea, where many examples show that the enzymatic activity involves structural dynamics. This is incorporated in the enzymatic mechanism while introducing several Michaelis-Menten pathways that are connected with fluctuating rates. Nevertheless, there is a mathematical relation connecting the behavior obtained from the basic Michaelis-Menten mechanism (that was indeed proved correct in many experiments) with the generalized Michaelis-Menten mechanisms involving dynamics and activity. This means that the measured activity of enzymes on the level of many enzymes may be explained with the simple Michaelis-Menten equation, yet, the actual activity of enzymes is richer and involves structural dynamics.

Source: wikipedia (June 2011)
**Stochastic chemical kinetics**

- Types of reaction described by **stochastic equations**:
  \[ E + S \xrightleftharpoons[k_2]{k_1} ES \xrightarrow{k_3} E + P \]

- **N** different types of molecules that randomly collide
  where \( X(t) = (x_1, \ldots, x_N) \) with \( x_i = \# \) molecules of sort \( i \)

- **Reaction probability** within infinitesimal interval \([t, t+\Delta]\):
  \[ \alpha_m(\vec{x}) \cdot \Delta = \Pr\{ \text{reaction } m \text{ in } [t, t+\Delta] \mid X(t) = \vec{x} \} \]
  where
  \[ \alpha_m(\vec{x}) = k_m \cdot \# \text{ possible combinations of reactant molecules in } \vec{x} \]

- This process is a **continuous-time Markov chain**.

**CTMCs are omnipresent!**

- Markovian queueing networks  
  (Kleinrock 1975)

- Stochastic Petri nets  
  (Molloy 1977)

- Stochastic activity networks  
  (Meyer & Sanders 1985)

- Stochastic process algebra  
  (Herzog et al., Hillston 1993)

- Probabilistic input/output automata  
  (Smolka et al. 1994)

- Calculi for biological systems  
  (Priami et al., Cardelli 2002)

**CTMCs are one of the most prominent models in performance analysis**

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**Enzyme-catalyzed substrate conversion as a CTMC**

**States:**

<table>
<thead>
<tr>
<th>States</th>
<th>init</th>
<th>goal</th>
</tr>
</thead>
<tbody>
<tr>
<td>enzymes</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>substrates</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>complex</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>products</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

**Transitions:**

- \( E + S \xrightleftharpoons[0.001]{1} C \xrightarrow[0.001 \times X_C]{1} E + P \)

  e.g., \((x_E, x_S, x_C, x_P) \xrightarrow{0.001 \times X_C} (x_E + 1, x_S, x_C - 1, x_P + 1)\) for \( x_C > 0 \)

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**Summary**

**Main points**

- Exponential distributions are closed under minimum.
- The probability to win a race amongst several exponential distributions only depends on their rates.
- A CTMC is a DTMC where state residence times are exponentially distributed.
- CTMC semantics distinguishes between enabledness and taking a transition.
- CTMCs are frequently used as semantical model for high-level formalisms.
Overview

Negative exponential distributions

What are continuous-time Markov chains?

Transient distribution

Timed reachability probabilities

Verifying continuous stochastic CTL

Verifying linear real-time properties

Transient distribution

Transient distribution of a CTMC

Transient state probability

Let $X(t)$ denote the state of a CTMC at time $t \in \mathbb{R}_{\geq 0}$. The probability to be in state $s$ at time $t$ is defined by:

$$ p_s(t) = \Pr\{X(t) = s\} = \sum_{s' \in S} \Pr\{X(0) = s'\} \cdot \Pr\{X(t) = s | X(0) = s'\} $$

Theorem: transient distribution as linear differential equation

The transient probability vector $p(t) = (p_s(t), \ldots, p_k(t))$ satisfies:

$$ p'(t) = p(t) \cdot (R - r) \quad \text{given} \quad p(0) $$

where $r$ is the diagonal matrix of vector $r$.

Computing transient probabilities

The transient probability vector $p(t) = (p_s(t), \ldots, p_k(t))$ satisfies:

$$ p'(t) = p(t) \cdot (R - r) \quad \text{given} \quad p(0) $$

Solution using standard knowledge yields:

$$ p(t) = p(0) \cdot e^{(R-r) \cdot t} $$

Computing a matrix exponential

First attempt: use Taylor-Maclaurin expansion. This yields

$$ p(t) = p(0) \cdot e^{(R-r) \cdot t} = p(0) \cdot \sum_{i=0}^{\infty} \frac{(R-r) \cdot t)^i}{i!} $$

But: numerical instability due to fill-in of $(R-r)^i$ in presence of positive and negative entries in the matrix $R-r$. 

Uniformization

Let CTMC $\mathcal{C} = (S, P, r, t_{\text{init}}, AP, L)$ with $S$ finite.

**Uniform CTMC**

CTMC $\mathcal{C}$ is uniform if $r(s) = r$ for all $s \in S$ for some $r \in \mathbb{R}_{>0}$.

[**Uniformization**](Gross and Miller, 1984)

Let $r \in \mathbb{R}_{>0}$ such that $r \geq \max_{s \in S} r(s)$. Then $\text{unif}(r, \mathcal{C})$ is the tuple $(S, \overline{P}, \tau, t_{\text{init}}, AP, L)$ with $\tau(s) = r$ for all $s \in S$, and:

$$\overline{P}(s, s') = \frac{r(s)}{r} P(s, s') \quad \text{if } s' \neq s \quad \text{and} \quad \overline{P}(s, s) = \frac{r(s)}{r} P(s, s) + 1 - \frac{r(s)}{r}.$$

It follows that $\overline{P}$ is a stochastic matrix and $\text{unif}(r, \mathcal{C})$ is a CTMC.

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Uniformization: example

**Uniformization**

Let $r \in \mathbb{R}_{>0}$ such that $r \geq \max_{s \in S} r(s)$. Then $\text{unif}(r, \mathcal{C}) = (S, \overline{P}, \tau, t_{\text{init}}, AP, L)$ with $\tau(s) = r$ for all $s \in S$, and:

$$\overline{P}(s, s') = \frac{r(s)}{r} P(s, s') \quad \text{if } s' \neq s \quad \text{and} \quad \overline{P}(s, s) = \frac{r(s)}{r} P(s, s) + 1 - \frac{r(s)}{r}.$$

CTMC $\mathcal{C}$ and its uniformized counterpart $\text{unif}(6, \mathcal{C})$.

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Strong bisimulation on DTMCs

**Probabilistic bisimulation** (Larsen & Skou, 1989)

Let $\mathcal{D} = (S, P, t_{\text{init}}, AP, L)$ be a DTMC and $R \subseteq S \times S$ an equivalence. Then: $R$ is a probabilistic bisimulation on $S$ if for any $(s, t) \in R$:

1. $L(s) = L(t)$, and
2. $P(s, c) = P(t, c)$ for all equivalence classes $C \in S/R$

where $P(s, c) = \sum_{s' \in C} P(s, s')$.

For states in $R$, the probability of moving by a single transition to some equivalence class is equal.

**Probabilistic bisimilarity**

Let $\mathcal{D}$ be a DTMC and $s, t$ states in $\mathcal{D}$. Then: $s$ is probabilistically bisimilar to $t$, denoted $s \sim_{\text{pr}} t$, if there exists a probabilistic bisimulation $R$ with $(s, t) \in R$. 
Weak bisimulation on DTMCs

Let $\mathcal{D} = (S, P, t_{\text{init}}, AP, L)$ be a DTMC and $R \subseteq S \times S$ an equivalence. Then: $R$ is a weak probabilistic bisimulation on $S$ if for any $(s, t) \in R$:

1. $L(s) = L(t)$, and
2. $P(s, [s]_{R}) < 1$ and $P(t, [t]_{R}) < 1$, then:
   \[
   \frac{P(s, C)}{1 - P(s, [s]_{R})} = \frac{P(t, C)}{1 - P(t, [t]_{R})}
   \]
   for all $C \in S/R$, $C \neq [s]_{R} = [t]_{R}$.
3. $s$ can reach a state outside $[s]_{R}$ iff $t$ can reach a state outside $[t]_{R}$.

Probabilistic weak bisimilarity

Let $\mathcal{D}$ be a DTMC and $s, t$ states in $\mathcal{D}$. Then: $s$ is probabilistically weak bisimilar to $t$, denoted $s \approx_p t$, if there exists a probabilistic weak bisimulation $R$ with $(s, t) \in R$.

Weak bisimulation on CTMCs

Probabilistic bisimulation [Buchholz, 1994]

Let $C = (S, P, r, t_{\text{init}}, AP, L)$ be a CTMC and $R \subseteq S \times S$ an equivalence. Then: $R$ is a probabilistic bisimulation on $S$ if for any $(s, t) \in R$:

1. $L(s) = L(t)$, and
2. $P(s, C) = P(t, C)$ for all equivalence classes $C \in S/R$.

The last two conditions amount to $R(s, C) = R(t, C)$ for all equivalence classes $C \in S/R$.

Probabilistic bisimilarity

Let $C$ be a CTMC and $s, t$ states in $C$. Then: $s$ is probabilistically bisimilar to $t$, denoted $s \sim_m t$, if there exists a probabilistic bisimulation $R$ with $(s, t) \in R$.
Reachability condition

**Remark**
Consider the following DTMC:

![DTMC diagram]

It is not difficult to establish \( s_1 \approx s_2 \). Note: \( P(s_1, [s_1]) = 1 \), but \( P(s_2, [s_2]) < 1 \). Both \( s_1 \) and \( s_2 \) can reach a state outside \( [s_1]_R = [s_2]_R \). The reachability condition is essential to establish \( s_1 \approx s_2 \) and cannot be dropped: otherwise \( s_1 \) and \( s_2 \) would be weakly bisimilar to an equally labelled absorbing state.

**A useful lemma**

Let \( C \) be a CTMC and \( R \) an equivalence relation on \( S \) with \( (s, t) \in R \). Then: the following two statements are equivalent:

1. If \( P(s, [s]) < 1 \) and \( P(t, [t]) < 1 \) then for all \( C \in S/R \), \( C \neq [s]_R = [t]_R \):
   
   \[
   \frac{P(s, C)}{1 - P(s, [s])} = \frac{P(t, C)}{1 - P(t, [t])} \quad \text{and} \quad R(s, S \setminus [s]) = R(t, S \setminus [t])
   \]

2. \( R(s, C) = R(t, C) \) for all \( C \in S/R \) with \( C \neq [s]_R = [t]_R \).

**Proof:**
Left as an exercise.

Weak bisimulation on CTMCs

**Weak probabilistic bisimulation**

Let \( C = (S, P, r, t_{init}, AP, L) \) be a CTMC and \( R \subseteq S \times S \) an equivalence. Then: \( R \) is a weak probabilistic bisimulation on \( S \) if for any \( (s, t) \in R \):

1. \( L(s) = L(t) \), and
2. \( R(s, C) = R(t, C) \) for all \( C \in S/R \) with \( C \neq [s]_R = [t]_R \)

**Weak probabilistic bisimilarity**

Let \( C \) be a CTMC and \( s, t \) states in \( C \). Then: \( s \) is weak probabilistically bisimilar to \( t \), denoted \( s \approx_m t \), if there exists a weak probabilistic bisimulation \( R \) with \( (s, t) \in R \).

Weak bisimulation on CTMCs: example

Equivalence relation \( R \) with \( S/R = \{ \{ s_1, s_2, s_3, s_4, s_5, s_6 \}, \{ u_1, u_2, u_3, u_4, u_5 \} \} \) is a weak bisimulation on the CTMC depicted above. This can be seen as follows. For \( C = \{ u_1, u_2, u_3, u_4, u_5 \} \), we have that all \( s \)-states enter \( C \) with rate 2. The rates between the \( s \)-states are not relevant.
Properties (without proof)

Strong and weak bisimulation in uniform CTMCs

For all uniform CTMCs $C$ and states $s, u$ in $C$, we have:

$$s \sim_m u \iff s \approx_m u \iff s \sim_p u.$$  

For any CTMC $C$, we have: $C \approx_m \text{unif}(r, C)$ with $r \geq \max_{s \in S} r(s)$.

Preservation of transient probabilities

For all CTMCs $C$ with states $s, u$ in $C$ and $t \in \mathbb{R}_{\geq 0}$, we have:

$$s \sim_m u \implies \varrho(t) = \varrho(t)$$

where $\varrho(0) = 1_s$ and $\varrho(0) = 1_u$ where $1_s$ is the characteristic function for state $s$, i.e., $1_s(s') = 1$ iff $s = s'$.

Computing transient probabilities

The transient probability vector $\varrho(t) = (\rho_0(t), \ldots, \rho_k(t))$ satisfies:

$$\varrho'(t) = \varrho(t) \cdot (R - r) \quad \text{given} \quad \varrho(0).$$

Standard knowledge yields: $\varrho(t) = \varrho(0) \cdot e^{(R - r) \cdot t}$.

As uniformization preserves transient probabilities, we replace $R - r$ by its variant for the uniformized CTMC, i.e., $\hat{R} - \hat{r}$. We have:

$$\hat{R}(s, s') = \hat{P}(s, s') \cdot r(s) = \hat{P}(s, s') \cdot r \quad \text{and} \quad r = 1 - r.$$

Thus:

$$\varrho(0) \cdot e^{(R - r) \cdot t} = \varrho(0) \cdot e^{(\hat{R} - \hat{r}) \cdot t} = \varrho(0) \cdot e^{(\hat{R} - 1) \cdot t} = \varrho(0) \cdot e^{-rt} \cdot e^{rt \hat{P}}.$$

Intermezzo: Poisson distribution

Poisson distribution

The Poisson distribution is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time $[0, t]$ if these events occur with a known average rate $r$ and independently of the time since the last event. Formally, the pdf is:

$$f(i; r, t) = e^{-r \cdot t} \left(\frac{(r \cdot t)^i}{i!}\right)$$

where $r$ is the mean of the Poisson distribution.

Remark

The Poisson distribution can be derived as a limiting case to the binomial distribution as the number of trials goes to infinity and the expected number of successes remains fixed.
Transient probabilities: example

Let initial distribution \( p(0) = (1,0) \), and time bound \( t=1 \). Then:

\[
p(1) \approx (0.404043, 0.595957)
\]

Truncating the infinite sum

 Computing transient probabilities

\[
p(t) = p(0) \cdot \sum_{i=0}^{\infty} e^{-r_i t} \frac{(r_i t)^i}{i!} P^i
\]

- Summation can be truncated \textit{a priori} for a given error bound \( \varepsilon > 0 \).
- The error that is introduced by truncating at summand \( k \varepsilon \) is:

\[
\left| \sum_{i=0}^{\infty} e^{-r_i t} \frac{(r_i t)^i}{i!} p(i) - \sum_{i=0}^{k \varepsilon} e^{-r_i t} \frac{(r_i t)^i}{i!} p(i) \right| \leq \varepsilon
\]

- Strategy: choose \( k \varepsilon \) minimal such that:

\[
\sum_{i=0}^{k \varepsilon+1} e^{-r_i t} \frac{(r_i t)^i}{i!} p(i) \leq \varepsilon
\]

Summary

Main points

- Bisimilar states are equally labelled and their cumulative rate to any equivalence class coincides.
- Weak bisimilar states have equal conditional probabilities to move to some equivalence class, and can either both leave their class or both can’t.
- Uniformization normalizes the exit rates of all states in a CTMC.
- Uniformization transforms a CTMC into a weak bisimilar one.
- Transient distribution are obtained by solving a system of linear differential equations.
- These equations can be solved conveniently on the uniformized CTMC.
Paths in a CTMC

Timed paths

Paths in CTMC $C$ are maximal (i.e., infinite) paths of alternating states and time instants:

$$\pi = s_0 \xrightarrow{t_0} s_1 \xrightarrow{t_1} s_2 \ldots$$

such that $s_i \in S$ and $t_i \in \mathbb{R}_{\geq 0}$. Let $Paths(C)$ be the set of paths in $C$ and $Paths^*(C)$ the set of finite prefixes thereof.

Time instant $t_i$ is the amount of time spent in state $s_i$.

Notations

- Let $\pi[i] := s_i$ denote the $(i+1)$-st state along the timed path $\pi$.
- Let $\pi(i) := t_i$ the time spent in state $s_i$.
- Let $\pi@t$ be the state occupied in $\pi$ at time $t \in \mathbb{R}_{\geq 0}$, i.e. $\pi@t := \pi[i]$ where $i$ is the smallest index such that $\sum_{j=0}^{i} \pi(j) > t$.

Probability measure on DTMCs

Cylinder set

Let $s_0, \ldots, s_k \in S$ with $Pr(s_i, s_{i+1}) > 0$ for $0 \leq i < k$ and $l_0, \ldots, l_{k-1}$ non-empty intervals in $\mathbb{R}_{\geq 0}$. The cylinder set of $s_0 l_0 l_1 \ldots l_{k-1} s_k$ is defined by:

$$Cyl(s_0, l_0, \ldots, l_{k-1}, s_k) = \{ \pi \in Paths(C) \mid \forall 0 \leq i \leq k. \pi[i] = s_i \text{ and } i < k \Rightarrow \pi(i) \in l_i \}$$

The cylinder set spanned by $s_0, l_0, \ldots, l_{k-1}, s_k$ thus consists of all infinite timed paths that have a prefix $\pi$ that lies in $s_0, l_0, \ldots, l_{k-1}, s_k$. Cylinder sets serve as basic events of the smallest $\sigma$-algebra on $Paths(C)$.

$\sigma$-algebra of a CTMC

The $\sigma$-algebra associated with CTMC $C$ is the smallest $\sigma$-algebra $F(Paths(s_0))$ that contains all cylinder sets $Cyl(s_0, l_0, \ldots, l_{k-1}, s_k)$ where $s_0 \ldots s_k$ is a path in the state graph of $C$ (starting in $s_0$) and $l_0, \ldots, l_{k-1}$ range over all sequences of non-empty intervals in $\mathbb{R}_{\geq 0}$.

Paths and probabilities

To reason quantitatively about the behavior of a CTMC, we need to define a probability space over its paths.

Intuition

For a given state $s$ in CTMC $C$:

- Sample space := set of all interval-timed paths $s_0 l_0 \ldots l_{k-1} s_k$ with $s = s_0$
- Events := sets of interval-timed paths starting in $s$
- Basic events := cylinder sets
- Cylinder set of finite interval-timed paths := set of all infinite timed paths with a prefix in the finite interval-timed path

Probability measure on CTMCs

Cylinder set

The cylinder set $Cyl(s_0, l_0, \ldots, l_{k-1}, s_k)$ of $s_0 l_0 \ldots l_{k-1} s_k$ is defined by:

$$\{ \pi \in Paths(C) \mid \forall 0 \leq i \leq k. \pi[i] = s_i \text{ and } i < k \Rightarrow \pi(i) \in l_i \}$$

Probability measure

$Pr$ is the unique probability measure on the $\sigma$-algebra $F(Paths(s_0))$ defined by induction on $k$ as follows: $Pr(Cyl(s_0)) = \delta_{init}(s_0)$ and for $k > 0$:

$$Pr(Cyl(s_0, l_0, \ldots, l_{k-1}, s_k)) = Pr(Cyl(s_0, l_0, \ldots, l_{k-2}, s_{k-1})) \cdot \int_{l_{k-1}} R(s_{k-1}, s_k) e^{-r_{s_{k-1}} t} dt.$$
Zeno theorem

Zeno path
Path $s_0 \xrightarrow{t_0} s_1 \xrightarrow{t_1} s_2 \xrightarrow{t_2} s_3 \ldots$ is called Zeno if $\sum_i t_i$ converges.

Intuition
In case $\sum_i t_i$ does not diverge, the timed path represents an "unrealistic" computation where infinitely many transitions are taken in a finite amount of time. Example:

$\quad s_0 \xrightarrow{\frac{1}{3}} s_1 \xrightarrow{\frac{1}{2}} s_2 \xrightarrow{\frac{1}{3}} s_3 \ldots$

In real-time systems, such executions are typically excluded from the analysis.

Thanks to the following theorem, Zeno paths do not harm for CTMCs.

Zeno theorem
For all states $s$ in any CTMC, $Pr\{ \pi \in \text{Paths}(s) \mid \pi \text{ is Zeno} \} = 0.$

---

Reachability events
Let CTMC $C$ with (possibly infinite) state space $S$.

(Simple) reachability
Eventually reach a state in $G \subseteq S$. Formally:

$\Diamond G = \{ \pi \in \text{Paths}(C) \mid \exists i \in \mathbb{N}. \pi[i] \in G \}$

Invariance, i.e., always stay in state in $G$:

$\Box G = \{ \pi \in \text{Paths}(C) \mid \forall i \in \mathbb{N}. \pi[i] \in G \} = \Diamond \overline{G}.$

Constrained reachability
Or "reach-avoid" properties where states in $F \subseteq S$ are forbidden:

$\overline{F} \cup G = \{ \pi \in \text{Paths}(C) \mid \exists i \in \mathbb{N}. \pi[i] \in G \land \forall j < i. \pi[j] \notin F \}$

---

Measurability

Measurability theorem
Events $\Diamond G, \Box G, \overline{F} \cup G, \Box \Diamond G$ and $\Diamond \overline{G}$ are measurable on any CTMC.

Proof:
Left as an exercise.
Reachability probabilities in finite CTMCs

Problem statement

Let $C$ be a CTMC with finite state space $S$, $s \in S$ and $G \subseteq S$.
Aim: determine $\Pr(s \models \diamondsuit G) = Pr_s(\diamondsuit G) = Pr_s\{\pi \in \text{Paths}(s) \mid \pi \models \diamondsuit G\}$
where $Pr_s$ is the probability measure in $C$ with single initial state $s$.

Characterisation of reachability probabilities

- Let variable $x_s = Pr(s \models \diamondsuit G)$ for any state $s$
  - if $G$ is not reachable from $s$, then $x_s = 0$
  - if $s \in G$ then $x_s = 1$
- For any state $s \in \text{Pre}^*(G) \setminus G$:
  $$x_s = \sum_{t \in S \setminus G \text{ reach } G \text{ via } t} P(s, t) \cdot x_t + \sum_{u \in G} P(s, u) \cdot \Pr_u(\text{reach } G \text{ in one step})$$

Timed reachability events

Let CTMC $C$ with (possibly infinite) state space $S$.

(Simple) timed reachability

Eventually reach a state in $G \subseteq S$ in the interval $I$. Formally:

$$\diamondsuit^I G = \{ \pi \in \text{Paths}(C) \mid \exists t \in I. \pi @ t \in G \}$$

Invariance, i.e., always stay in state in $G$ in the interval $I$:

$$\square^I G = \{ \pi \in \text{Paths}(C) \mid \forall t \in I. \pi @ t \in G \} = \diamondsuit^I \neg G.$$ 

Constrained timed reachability

Or "reach-avoid" properties where states in $F \subseteq S$ are forbidden:

$$\text{F U}^I G = \{ \pi \in \text{Paths}(C) \mid \exists t \in I. \pi @ t \in G \land \forall d < t. \pi @ d \notin F \}$$

Verifying CTMCs

Verifying untimed properties

So, computing reachability probabilities is exactly the same as for DTMCs. The same holds for constrained reachability, persistence and repeated reachability. In fact, all PCTL and LTL formulas can be checked on the embedded DTMC $(S, P, \iota_{\text{init}}, AP, L)$ using the techniques described before in these lecture slides.

Justification:

As the above temporal logic formulas or events do not refer to elapsed time, it is not surprising that they can be checked on the embedded DTMC.

Measurability

Measurability theorem

Events $\diamondsuit^I G$, $\square^I G$, and $\text{F U}^I G$ are measurable on any CTMC.

Proof:

Left as an exercise.
## Timed reachability probabilities in finite CTMCs

### Problem statement
Let $C$ be a CTMC with finite state space $S$, $s \in S$, $t \in \mathbb{R}_{>0}$ and $G \subseteq S$. Aim: $Pr(s \models \diamond_{\leq t} G) = Pr_s(\text{Paths}(s) \mid s \models \diamond_{\leq t} G)$ where $Pr_s$ is the probability measure in $C$ with single initial state $s$.

### Characterisation of timed reachability probabilities
- Let function $x_s(t) = Pr(s \models \diamond_{\leq t} G)$ for any state $s$
  - if $G$ is not reachable from $s$, then $x_s(t) = 0$ for all $t$
  - if $s \in G$, then $x_s(t) = 1$ for all $t$
- For any state $s \in \text{Pre}^*(G) \setminus G$:
  \[
  x_s(t) = \int_0^t \sum_{s' \in S} \mathbf{R}(s, s') \cdot e^{-r(s) \cdot x} \cdot x_s'(t-x) \, dx
  \]

### Timed reachability probabilities $=$ transient probabilities

#### Aim
Compute $Pr(s \models \diamond_{\leq t} G)$ in CTMC $C$. Observe that once a path $\pi$ reaches $G$ within $t$ time, then the remaining behaviour along $\pi$ is not important. This suggests to make all states in $G$ absorbing.

Let CTMC $C = (S, P, r, t_{\text{init}}, AP, L)$ and $G \subseteq S$. The CTMC $C[G] = (S, P_G, r, t_{\text{init}}, AP, L)$ with $P_G(s, t) = P(s, t)$ if $s \not\in G$ and $P_G(s, t) = 1$ if $s \in G$.

All outgoing transitions of $s \in G$ are replaced by a single self-loop at $s$.

#### Lemma
\[
Pr(s \models \diamond_{\leq t} G) = Pr(s \models \Diamond_{\leq t} G) = \frac{p(t)}{1} \text{ with } p(0) = 1.
\]

### Reachability

#### Reachability probabilities in finite DTMCs and CTMCs
Can be obtained by solving a system of linear equations for which many efficient techniques exist.

#### Timed reachability probabilities in finite CTMCs
Can be obtained by solving a system of Volterra integral equations. This is in general a non-trivial issue, inefficient, and has several pitfalls such as numerical stability.

#### Solution
Reduce the problem of computing $Pr(s \models \diamond_{\leq t} G)$ to an alternative problem for which well-known efficient techniques exist: computing transient probabilities (see previous lecture).

### Constrained timed reachability probabilities

#### Problem statement
Let $C$ be a CTMC with finite state space $S$, $s \in S$, $t \in \mathbb{R}_{>0}$ and $G, F \subseteq S$. Aim: $Pr(s \models F U_{\leq t} G) = Pr_s(\text{Paths}(s) \mid s \models F U_{\leq t} G) = Pr_s(\text{Paths}(s) \mid s \models F U_{\leq t} G)$.

#### Characterisation of timed reachability probabilities
- Let function $x_s(t) = Pr(s \models F U_{\leq t} G)$ for any state $s$
  - if $G$ is not reachable from $s$ via $F$, then $x_s(t) = 0$ for all $t$
  - if $s \in G$, then $x_s(t) = 1$ for all $t$
- For any state $s \in \text{Pre}^*(G) \setminus (F \cup G)$:
  \[
  x_s(t) = \int_0^t \sum_{s' \in S} \mathbf{R}(s, s') \cdot e^{-r(s) \cdot x} \cdot x_s'(t-x) \, dx
  \]
Constrained timed reachability = transient probabilities

Aim
Compute \( P_r(s \models F U \leq t \mid G) \) in CTMC \( C \). Observe (as before) that once a path \( \pi \) reaches \( G \) within time \( t \) via \( F \), then the remaining behaviour along \( \pi \) is not important. Now also observe that once \( s \in F \setminus G \) is reached within time \( t \), then the remaining behaviour along \( \pi \) is not important. This suggests to make all states in \( G \) and \( F \setminus G \) absorbing.

Lemma
\[
\begin{align*}
\Pr(s \models F U \leq t \mid G) &= \Pr(s \models \Diamond \leq t \mid G) = \mathbf{p}(t) \text{ with } \mathbf{p}(0) = \mathbf{1}_G.
\end{align*}
\]

Summary
Main points
- Cylinder sets in a CTMC are paths that share interval-timed path prefixes.
- Reachability, persistence and repeated reachability can be checked as on DTMCs.
- Timed reachability probabilities can be characterised as Volterra integral equation system.
- Computing timed reachability probabilities can be reduced to transient probabilities.
- Weak and strong bisimulation preserves timed reachability probabilities.

Strong and weak bisimulation

Bisimulation preserves timed reachability events
Let \( C \) be a CTMC with state space \( S \), \( s, u \in S \), \( t \in \mathbb{R}_{\geq 0} \) and \( G, F \subseteq S \). Then:
\[
\begin{align*}
1. \ s \sim_m u \text{ implies } & P_r(s \models F U \leq t \mid G) = P_r(u \models F U \leq t \mid G) \\
2. \ s \approx_m u \text{ implies } & P_r(s \models F U \leq t \mid G) = P_r(u \models F U \leq t \mid G)
\end{align*}
\]
provided \( F \) and \( G \) are closed under \( \sim_m \) and \( \approx_m \), respectively.

Proof:
Left as an exercise.

Overview
- Negative exponential distributions
- What are continuous-time Markov chains?
- Transient distribution
- Timed reachability probabilities
- Verifying continuous stochastic CTL
- Verifying linear real-time properties
Continuous Stochastic Logic

- CSL is a language for formally specifying properties over CTMCs.
- It is a branching-time temporal logic based on CTL.
- Formula interpretation is Boolean, i.e., a state satisfies a formula or not.
- Like in PCTL, the main operator is $P_J(\varphi)$
  - where $\varphi$ constrains the set of paths and $J$ is a threshold on the probability.
  - it is the probabilistic counterpart of $\exists$ and $\forall$ path-quantifiers in CTL.
- The new features are a timed version of the next and until-operator.
  - $\circ^I \Phi$ asserts that a transition to a $\Phi$-state can be made at time $t \in I$.
  - $\Phi U^I \Psi$ asserts that a $\Psi$-state can be reached via $\Phi$-states at time $t \in I$.

CSL syntax

[Baier, Katoen & Hermanns, 1999]

Continuous Stochastic Logic: Syntax

CSL consists of state- and path-formulas.
- CSL state formulas over the set $AP$ obey the grammar:
  \[ \Phi ::= \text{true} \mid a \mid \Phi_1 \land \Phi_2 \mid \neg \Phi \mid P_J(\varphi) \]
  where $a \in AP$, $\varphi$ is a path formula and $J \subseteq [0,1]$, $J \neq \emptyset$ is a non-empty interval.

- CSL path formulae are formed according to the following grammar:
  \[ \varphi ::= \circ^I \Phi \mid \Phi_1 U^I \Phi_2 \]
  where $\Phi$, $\Phi_1$, and $\Phi_2$ are state formulae and $I \subseteq \mathbb{R}_{\geq 0}$ an interval.

Abbreviate $P_{[0,0.5]}(\varphi)$ by $P_{\leq 0.5}(\varphi)$ and $P_{[0,1]}(\varphi)$ by $P_{>0}(\varphi)$.

Derived operators

\[ \boxdot \Phi = \text{true}U \Phi \]
\[ \boxdot^I \Phi = \text{true}U^I \Phi \]
\[ P_{\leq p}(\Box \Phi) = P_{>1-p}(\Diamond \neg \Phi) \]
\[ P_{(p,q)}(\Box^I \Phi) = P_{[1-q,1-p]}(\Diamond^I \neg \Phi) \]
**Paths in a CTMC**

### Timed paths

*Paths* in CTMC $C$ are maximal (i.e., infinite) paths of alternating states and time instants:

$$\pi = s_0 \xrightarrow{a} s_1 \xrightarrow{b} s_2 \cdots$$

such that $s_i \in S$ and $t_i \in \mathbb{R}_{\geq 0}$. Let $Paths(C)$ be the set of paths in $C$ and $Paths^*(C)$ the set of finite prefixes thereof.

### Notations

- Let $\pi[i]$ denote the $(i+1)$-st state along the timed path $\pi$.
- Let $\pi(i)$ denote the time spent in state $s_i$.
- Let $\pi@t$ be the state occupied in $\pi$ at time $t \in \mathbb{R}_{\geq 0}$, i.e. $\pi@t := \pi[i]$ where $i$ is the smallest index such that $\sum_{j=0}^{i} \pi(j) > t$.

### CSL semantics (1)

**Notation**

$C, s \models \Phi$ if and only if state-formula $\Phi$ holds in state $s$ of CTMC $C$.

**Satisfaction relation for state formulas**

The satisfaction relation $\models$ is defined for CSL state formulas by:

- $s \models a$ iff $a \in L(s)$
- $s \models \neg \Phi$ iff $s \not\models \Phi$
- $s \models \Phi \land \Psi$ iff $(s \models \Phi)$ and $(s \models \Psi)$
- $s \models Pr_j(\varphi)$ iff $Pr(s \models \varphi) \in J$

where $Pr(s \models \varphi) = Pr_s\{ \pi \in Paths(s) \mid \pi \models \varphi \}$.

This is as for PCTL, except that $Pr$ is the probability measures on cylinder sets of timed paths in CTMC $C$.

### Example properties

- Transient probabilities to be in *goal* state at time point 4:
  $$P_{\geq 0.92}(\diamond = 4 \text{ goal})$$
- With probability $\geq 0.92$, a goal state is reached legally:
  $$P_{\geq 0.92}(\neg \text{ illegal } U \text{ goal})$$
- . . . in maximally 137 time units:
  $$P_{\geq 0.92}(\neg \text{ illegal } U \leq 137 \text{ goal})$$
- . . . once there, remain there almost surely for the next 31 time units:
  $$P_{\geq 0.92}(\neg \text{ illegal } U \leq 137 P_{\geq 0.92}(\exists t \in [0, 31] \text{ goal}))$$

### CSL semantics (2)

**Satisfaction relation for path formulas**

Let $\pi = s_0 t_0 s_1 t_1 s_2 \ldots$ be an infinite path in CTMC $C$.

The satisfaction relation $\models$ is defined for state formulas by:

- $\pi \models \diamond^i \Phi$ iff $s_1 \models \Phi \land t_0 \not\in I$
- $\pi \models \Phi U \Psi$ iff $\exists t \in I. ((\forall t' \in [0, t). \pi@t' \models \Phi) \land \pi@t \models \Psi)$

**Standard next- and until-operators**

- $X\Phi \equiv \diamond^1 \Phi$ with $I = \mathbb{R}_{\geq 0}$
- $\Phi U \Psi \equiv \Phi U[0, \infty) \Psi$ with $I = \mathbb{R}_{\geq 0}$
Verifying Continuous-Time Markov Chains

Verifying continuous stochastic CTL

Measurability

**CSL measurability**

For any CSL path formula \( \varphi \) and state \( s \) of CTMC \( C \), the set \( \{ \pi \in \text{Paths}(s) \mid \pi \models \varphi \} \) is measurable.

**Proof:**

Rather straightforward; left as an exercise.

CSL model checking

**CSL model checking problem**

- **Input:** a finite CTMC \( C = (S, P, r, \ell_{\text{init}}, AP, L) \), state \( s \in S \), and CSL state formula \( \Phi \)
- **Output:** yes, if \( s \models \Phi \); no, otherwise.

**Basic algorithm**

In order to check whether \( s \models \Phi \) do:

1. Compute the satisfaction set \( \text{Sat}(\Phi) = \{ s \in S \mid s \models \Phi \} \).
2. This is done recursively by a bottom-up traversal of \( \Phi \)'s parse tree.
   - The nodes of the parse tree represent the subformulae of \( \Phi \).
   - For each node, i.e., for each subformula \( \Psi \) of \( \Phi \), determine \( \text{Sat}(\Psi) \).
   - Determine \( \text{Sat}(\Psi) \) as function of the satisfaction sets of its children:
     \( \text{e.g.}, \text{Sat}(\psi_1 \land \psi_2) = \text{Sat}(\psi_1) \cap \text{Sat}(\psi_2) \) and \( \text{Sat}(\neg \psi) = S \setminus \text{Sat}(\psi) \).
3. Check whether state \( s \) belongs to \( \text{Sat}(\Phi) \).

Core model checking algorithm

**Probabilistic operator \( \mathbb{P} \)**

In order to determine whether \( s \in \text{Sat}(\mathbb{P}(\varphi)) \), the probability \( Pr(s \models \varphi) \) for the event specified by \( \varphi \) needs to be established. Then

\[
\text{Sat}(\mathbb{P}(\varphi)) = \{ s \in S \mid Pr(s \models \varphi) \in J \}.
\]

Let us consider the computation of \( Pr(s \models \varphi) \) for all possible \( \varphi \).

The next-step operator

Recall that: \( s \models \mathbb{P}(\Box' \Phi) \) if and only if \( Pr(s \models \Box' \Phi) \in J \).

**Lemma**

\[
Pr(s \models \Box' \Phi) = \frac{e^{-r(s) \cdot \inf I} - e^{-r(s) \cdot \sup I}}{\text{probability to leave } s \text{ in interval } I} \cdot \sum_{s' \in \text{Sat}(\Phi)} P(s, s').
\]

**Algorithm**

Considering the above equation for all states simultaneously yields:

\[
(Pr(s \models \Box' \Phi))_{s \in S} = b_I^T \cdot P
\]

with \( b_I \) is defined by \( b_I(s) = e^{-r(s) \cdot \inf I} - e^{-r(s) \cdot \sup I} \) if \( s \in \text{Sat}(\Phi) \) and 0 otherwise, and \( b_I^T \) is the transposed variant of \( b_I \).
Time-bounded until (1)

Recall that: $s \models P_j(\Phi \land F \Psi)$ if and only if $Pr(s \models \Phi \land F \Psi) \in J$.

**Lemma**

Let $S_{-1} = \text{Sat}(\Psi)$. $S_{>0} = S \setminus (\text{Sat}(\Phi) \cup \text{Sat}(\Psi))$, and $S = S \setminus (S_{>0} \cup S_{-1})$. Then:

$$Pr(s \models \Phi \land F \Psi) = \begin{cases} 1 & \text{if } s \in S_{-1} \\ 0 & \text{if } s \in S_{>0} \\ \int_0^\infty \sum_{s' \in S} R(s, s') \cdot e^{-r(s)x} \cdot Pr(s' \models \Phi \land F \Psi) \, dx & \text{otherwise} \end{cases}$$

This is a slight generalisation of the Volterra integral equation system for timed reachability.

**Algorithm for checking $Pr(s \models \Phi \land F \Psi) \in J$**

1. If $t = \infty$, then use approach for until (as in PCTL): solve a system of linear equations.
2. Determine recursively $\text{Sat}(\Phi)$ and $\text{Sat}(\Psi)$.
3. Make all states in $S \setminus \text{Sat}(\Phi)$ and $\text{Sat}(\Psi)$ absorbing.
4. Uniformize the resulting CTMC with respect to its maximal rate.
5. Determine the transient probability at time $t$ using $s$ as initial distribution.
6. Return yes if transient probability of all $\Psi$-states lies in $J$, and no otherwise.

Time-bounded until (2)

Let $S_{-1} = \text{Sat}(\Psi)$, $S_{>0} = S \setminus (\text{Sat}(\Phi) \cup \text{Sat}(\Psi))$, and $S = S \setminus (S_{>0} \cup S_{-1})$. Then:

$$Pr(s \models \Phi \land F \Psi) = \begin{cases} 1 & \text{if } s \in S_{-1} \\ 0 & \text{if } s \in S_{>0} \\ \int_0^\infty \sum_{s' \in S} R(s, s') \cdot e^{-r(s)x} \cdot Pr(s' \models \Phi \land F \Psi) \, dx & \text{otherwise} \end{cases}$$

Recall that

$$Pr(s \models \Phi \land F \Psi) = \begin{cases} 1 & \text{if } s \in S_{-1} \\ 0 & \text{if } s \in S_{>0} \\ \int_0^\infty R(s, s') \cdot e^{-r(s)x} \cdot Pr(s' \models \Phi \land F \Psi) \, dx & \text{otherwise} \end{cases}$$

Phrased using CSL state formulas

$$Pr(s \models \Phi \land F \Psi) = \begin{cases} 1 & \text{in } C[\text{Sat}(\Phi \cup \text{Sat}(\Psi)) \cup \text{Sat}(\Psi)] \\ 0 & \text{in } C[\text{Sat}(\Phi \cup \text{Sat}(\Psi)) \cup \text{Sat}(\Psi)] \end{cases}$$

Time-bounded until (3)

possible optimizations:

1. Make all states in $S \setminus \text{Sat}(\exists (\Phi \land F \Psi))$ absorbing.
2. Make all states in $\text{Sat}(\forall (\Phi \land F \Psi))$ absorbing.
3. Replace the labels of all states in $S \setminus \text{Sat}(\exists (\Phi \land F \Psi))$ by unique label zero.
4. Replace the labels of all states in $\text{Sat}(\forall (\Phi \land F \Psi))$ by unique label one.
5. Perform bisimulation minimization on all states.

The last step collapses all states in $S \setminus \text{Sat}(\exists (\Phi \land F \Psi))$ into a single state, and does the same with all states in $\text{Sat}(\forall (\Phi \land F \Psi))$. 

Time-bounded until (4)
Verifying Continuous-Time Markov Chains

Preservation of CSL-formulas

**Bisimulation and CSL-equivalence coincide**

Let \( C \) be a finitely branching CTMC and \( s, t \) states in \( C \). Then:

\[
s \sim_m t \text{ if and only if } s \text{ and } t \text{ are CSL-equivalent.}
\]

**Remarks**

If for CSL-formula \( \Phi \) we have \( s \models \Phi \) but \( t \not\models \Phi \), then it follows \( s \not\sim_m t \). A single CSL-formula suffices!

---

**Uniformization and CSL**

**Uniformization and CSL**

For any finite CTMC \( C \) with state space \( S \), \( r \geq \max\{ r(s) \mid s \in S \} \) and \( \Phi \) a CSL-without-next-formula:

\[
\text{Sat}^C(\Phi) = \text{Sat}^{C'}(\Phi) \text{ where } C' = \text{unif}(r, C).
\]

**Uniformization and CSL**

For any uniformized CTMC: CSL-equivalence coincides with CSL-without-next-equivalence.

---

**Weak bisimulation and CSL-without-next-equivalence coincide**

Let \( C \) be a finitely branching CTMC and \( s, t \) states in \( C \). Then:

\[
s \approx_m t \text{ if and only if } s \text{ and } t \text{ are CSL-without-next-equivalent.}
\]

Here, CSL-without-next is the fragment of CSL where the next-operator \( \circ \) does not occur.

**Remarks**

If for CSL-without-next-formula \( \Phi \) we have \( s \models \Phi \) but \( t \not\models \Phi \), then it follows \( s \not\approx_m t \).

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**Time complexity**

Let \( |\Phi| \) be the size of \( \Phi \), i.e., the number of logical and temporal operators in \( \Phi \).

**Time complexity of CSL model checking**

For finite CTMC \( C \) and CSL state-formula \( \Phi \), the CSL model-checking problem can be solved in time

\[
\mathcal{O}(\text{poly(size}(C)) \cdot t_{\max} \cdot |\Phi|)
\]

where \( t_{\max} = \max\{ t \mid \Psi_1 \cup t\Psi_2 \text{ occurs in } \Phi \} \) with and \( t_{\max} = 1 \) if \( \Phi \) does not contain a time-bounded until-operator.
Some practical verification times

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{chart.png}
\caption{Verification time vs. state space size for different models.}
\end{figure}

- Command-line tool MRMC ran on a Pentium 4, 2.66 GHz, 1 GB RAM laptop.
- CSL formulas are time-bounded until-formulas.

Summary

- CSL is a variant of PCTL with timed next and timed until.
- Sets of paths fulfilling CSL path-formula \( \varphi \) are measurable.
- CSL model checking is performed by a recursive descent over \( \Phi \).
- The timed next operator amounts to a single vector-matrix multiplication.
- The time-bounded until-operator \( U^{\leq t} \) is solved by uniformization.
- The worst-case time complexity is polynomial in the size of the CTMC and linear in the size of the formula.

Overview

- Negative exponential distributions
- What are continuous-time Markov chains?
- Transient distribution
- Timed reachability probabilities
- Verifying continuous stochastic CTL
- Verifying linear real-time properties

Robot navigation

- The robot randomly moves through the cells, and resides in a cell for an exponentially distributed amount of time.
- Gray cells are dangerous; the robot should leave them quickly.
- Property: What is the probability to reach \( B \) from \( A \) within 10 time units while residing in any dangerous zone for at most 2 time units?
**Robot navigation: property**

**Property:**
What is the probability to reach $B$ from $A$ within 10 time units while residing in any dangerous zone for at most 2 time units?

**Deterministic timed automata**

A Deterministic Timed Automaton (DTA) $A$ is a tuple $(\Sigma, X, Q, q_0, F, \rightarrow)$:

- $\Sigma$ - alphabet
- $X$ - finite set of clocks
- $Q$ - finite set of locations
- $q_0 \in Q$ - initial location
- $F \subseteq Q$ - accept locations
- $\rightarrow \in Q \times \Sigma \times C(X) \times 2^X \times Q$ - transition relation;

**Determinism:** $q \xrightarrow{a, \xi, X} q'$ and $q \xrightarrow{a, \xi', X'} q''$ implies $\xi \cap \xi' = \emptyset$

**Model checking Markov chains**

<table>
<thead>
<tr>
<th></th>
<th>branching time</th>
<th>linear time</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>discrete-time</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(DTMC $D$)</td>
<td>PCTL</td>
<td>LTL</td>
</tr>
<tr>
<td></td>
<td>linear equations</td>
<td>automata-based</td>
</tr>
<tr>
<td>(DTMC $D$)</td>
<td>[HJ94] (+)</td>
<td>[V85,CSS03] (**+)</td>
</tr>
<tr>
<td></td>
<td>PTIME</td>
<td>PSPACE-C</td>
</tr>
<tr>
<td><strong>continuous-time</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(CTMC $C$)</td>
<td>untimed PCTL</td>
<td>untimed LTL</td>
</tr>
<tr>
<td></td>
<td>$emb(C)$ (+)</td>
<td>$emb(C)$ (**+)</td>
</tr>
<tr>
<td></td>
<td>PTIME</td>
<td>PSPACE-C</td>
</tr>
</tbody>
</table>
What are we interested in?

Problem statement:
Given model CTMC \( C \) and specification DTA \( A \), determine the fraction of runs in \( C \) that satisfy \( A \):

\[
Pr(C \models A) := Pr\{\text{Paths in } C \text{ accepted by } A\}
\]
**Product construction**

CTMC $C$ with state space $S$

\[ S_0 \xrightarrow{L(s_0)=A_0} S_1 \xrightarrow{L(s_1)=A_1} S_2 \xrightarrow{L(s_2)=A_2} \ldots \]

\[ S_n \xrightarrow{L(s_n)=A_n} q_n \xrightarrow{q_{n+1}} \]

DTA $A$ with state space $Q$

\[ q_0 \in Q_0 \xrightarrow{A_0} q_1 \xrightarrow{A_1} q_2 \xrightarrow{A_2} \ldots \]

\[ q_n \xrightarrow{A_n} q_{n+1} \]

\[ S \times Q \]

\[ \pi \]

**Product construction: example**

An example CTMC $C$ (left) and DTA $A$ (right)

\[ a, x < 1, \varnothing \]

\[ b, x > 1, \varnothing \]

\[ \{a\} \xrightarrow{0.5} \{a\} \xrightarrow{0.3} \{a\} \xrightarrow{1} \{b\} \xrightarrow{1} q_1 \]

\[ a, 1 < x < 2, \{x\} \]

\[ b, x > 1, \varnothing \]

\[ \{a\} \xrightarrow{0.5} \{a\} \xrightarrow{0.3} \{a\} \xrightarrow{1} \{b\} \xrightarrow{1} q_1 \]

\[ a, 1 < x < 2, \{x\} \]

\[ b, x > 1, \varnothing \]

An example CTMC $C$ (left up) and DTA $A$ (right up) and $C \otimes RG(A)$ (below)

\[ a, x < 1, \varnothing \]

\[ \{a\} \xrightarrow{0.5} \{a\} \xrightarrow{0.3} \{a\} \xrightarrow{1} \{b\} \xrightarrow{1} q_1 \]

\[ a, 1 < x < 2, \{x\} \]

\[ b, x > 1, \varnothing \]

\[ \{a\} \xrightarrow{0.5} \{a\} \xrightarrow{0.3} \{a\} \xrightarrow{1} \{b\} \xrightarrow{1} q_1 \]

\[ a, 1 < x < 2, \{x\} \]

\[ b, x > 1, \varnothing \]

**One-clock DTA: partitioning $C \otimes RG(A)$**

- constants $c_0 < \ldots < c_m$ in $A$ yields $m+1$ subgraphs.
- subgraph $i$ captures behaviour of $C$ and $A$ in $[c_i, c_{i+1})$.
- any subgraph is a CTMC, resets lead to subgraph 0, delays to $i+1$.
- a subgraph with its resets yields an “augmented” CTMC.
**One-clock DTA: partitioning** \( C \otimes RG(A) \)

\[ \begin{array}{cccc}
\text{(a) } C_0 & \text{(b) } C_1 & \text{(c) } C_1' & \text{(d) } C_2 \\
\end{array} \]

**Theorem**

For CTMC \( C \) with initial distribution \( \alpha \), 1-clock DTA \( A \) we have that:

\[
Pr(C \models A) = \alpha \cdot u
\]

where \( u \) is the solution of the linear equation system \( x \cdot M = f \), with

\[
M = \begin{pmatrix}
I_n - B^{m-1} & A^{m-1}
\end{pmatrix}
\]

and \( f \) is the characterizing vector of the final states in subgraph \( m \), and \( A \) and \( B \) are obtained from transient probabilities in all subgraphs.

**Reachability in (our) PDPs**

- For **single-clock** DTA, reachability probabilities in (our) PDPs are characterized by the least solution of a linear equation system, whose coefficients are solutions of some ordinary differential equations (ODEs).
- For these coefficients either an analytical solution (for small state space) can be obtained or an arbitrarily closely approximated solution can be determined efficiently.
- In **multi-clock** DTA, reachability probabilities in (our) PDPs are characterized as the least solution of a Volterra integral equation system of the second type.
- This solution can be approximated by solving a system of partial differential equations (PDEs).
Robot navigation revisited

Black squares are walls. The residence time in consecutive C-cells $< T_1$. The residence time in consecutive D-cells $< T_2$.

Systems biology: immune-receptor signaling

$\text{Migands can react with a receptor } R \text{ with rate } k_{+1} \text{ yielding a ligand-receptor LR}$

$\text{LR undergoes a sequence of } N \text{ modifications with a constant rate } k_p \text{ yielding } B_1, \ldots, B_N$ 

$\text{LR } B_N \text{ can link with an inactive messenger with rate } k_{+x} \text{ yielding a ligand-receptor-messenger (LRM)}$.

$\text{The LRM decomposes into an active messenger with rate } k_{cat}$
Verification results

<table>
<thead>
<tr>
<th>M</th>
<th>#CTMC states</th>
<th>No lumping</th>
<th>With lumping</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>#&lt;i&gt;⊗&lt;/i&gt; states</td>
<td>time(s)</td>
<td>#blocks</td>
</tr>
<tr>
<td>1</td>
<td>18</td>
<td>31</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>150</td>
<td>203</td>
<td>0.06</td>
</tr>
<tr>
<td>3</td>
<td>774</td>
<td>837</td>
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<tr>
<td>4</td>
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<td>2731</td>
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<tr>
<td>5</td>
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<td>7579</td>
<td>152.54</td>
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<tr>
<td>6</td>
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<td>18643</td>
<td>1547.45</td>
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<tr>
<td>7</td>
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<tr>
<td>8</td>
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<td>86656</td>
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<tr>
<td>9</td>
<td>336049</td>
<td>169024</td>
<td>71079.15</td>
</tr>
<tr>
<td>10</td>
<td>675817</td>
<td>312882</td>
<td>205552.36</td>
</tr>
</tbody>
</table>

In the case of no lumping, 99% of time is spent on transient analysis.

Multi-multi-core model checking

<table>
<thead>
<tr>
<th>N</th>
<th>4 Cores</th>
<th>20 Cores</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>time(s)</td>
<td>speedup</td>
</tr>
<tr>
<td>3</td>
<td>0.45</td>
<td>3.03</td>
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<tr>
<td>4</td>
<td>5.3</td>
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<tr>
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<tr>
<td>10</td>
<td>70623.46</td>
<td>2.91</td>
</tr>
</tbody>
</table>

Parallelization of the transient analysis only; not the lumping.

Summary

Take-home messages

- Checking CTMCs against deterministic timed automata (DTA).
- Efficient numerical algorithm for one-clock DTA:
  - using **standard** means: region construction, graph analysis, transient analysis, linear equation systems.
  - **three orders** of magnitude faster than alternative approaches.
  - **natural support** for parallelization and bisimulation minimization.
- Discretization approach for multiple-clock DTA with error bounds.