A Survey of Program Termination: Practical and Theoretical Challenges

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VTSA 2014
Luxembourg, October 2014

Termination of Linear Programs

$\mathbf{x} := \mathbf{a};$
while $\mathbf{u} \cdot \mathbf{x} \geq 0$ do
$\quad \mathbf{x} := \mathbf{M} \cdot \mathbf{x};$

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Termination Problem

Instance: $\langle\ \mathbf{a};\ \mathbf{u};\ \mathbf{M}\ \rangle$ over $\mathbb{Z}$ or $\mathbb{Q}$
Question: Does this program terminate?

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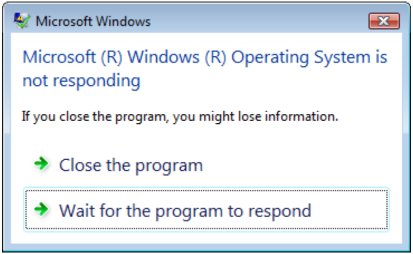
Much work on this and related problems in the literature over the last three decades:

- Manna, Pnueli, Kannan, Lipton, Sagiv, Podelski, Rybalchenko, Cook, Dershowitz, Tiwari, Braverman, Kovács, Ben-Amram, Genaim, ...
- Approaches include:
  - linear ranking functions
  - size-change termination methods
  - spectral techniques
  - ...
- Tools include:

TERMINATOR

proof tools for termination and liveness

ACL2


Microsoft Windows
Microsoft (R) Windows (R) Operating System is not responding
If you close the program, you might lose information.
Close the program
Wait for the program to respond

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Theorem

If $\mathbf{M}$ *has dimension* $5{\times}5$ *or less, Termination is decidable.*

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Theorem

If $\mathbf{M}$ is diagonalisable and has dimension 9×9 or less, Termination is decidable.

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Markov Chain Problem

Instance: $\langle$ stochastic matrix $\mathbf{M}$; $r \in (0,1]$ $\rangle$

Question: Does $\exists T$ s.t. $\forall n \geq T$, $(1,0,\ldots,0) \cdot \mathbf{M}^n \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \geq r$?

Positivity and Zeros of Linear Recurrence Sequences

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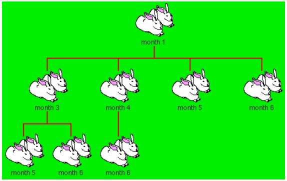

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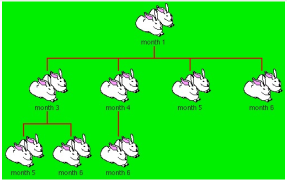

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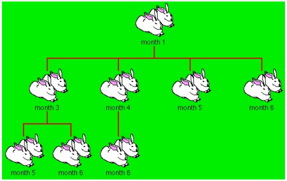

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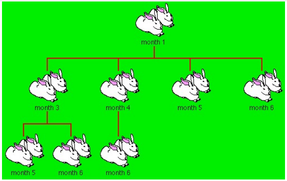

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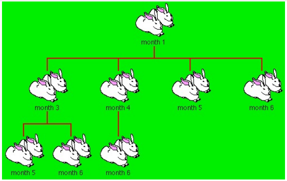

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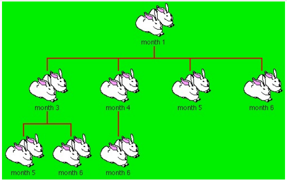

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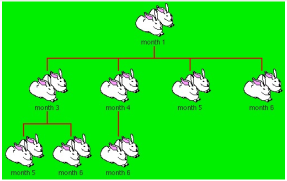

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Skolem Problem

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Question: Does $\exists n$ such that $u_n = 0$?

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- Fibonacci has **order** 2 $\Longleftrightarrow$ matrix $\mathbf{M}$ has **dimension** 2×2
- Fibonacci sequence is **simple** $\Longleftrightarrow$ $\mathbf{M}$ is **diagonalisable**

Linear Recurrence Sequences

- Numbers $\langle u_0, u_1, u_2, \ldots \rangle$ form a **linear recurrence sequence** if there exist k and constants $a_1, \ldots, a_k$, such that

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- The linear recurrence sequence is **simple** if its characteristic polynomial has no repeated roots
- Let $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{C}$ be the characteristic roots. There exist polynomials $p_1(x), p_2(x), \ldots, p_m(x) \in \mathbb{C}[x]$ such that

$$u_n = p_1(n)\lambda_1^n + p_2(n)\lambda_2^n + \ldots + p_m(n)\lambda_m^n$$

  In general $\lambda_1, \ldots, \lambda_k$ and all coefficients of $p_1(x), \ldots, p_m(x)$ are algebraic numbers

Linear Recurrence Sequences

- Numbers $\langle u_0, u_1, u_2, \ldots \rangle$ form a **linear recurrence sequence** if there exist k and constants $a_1, \ldots, a_k$, such that

$$\forall n \geq 0, \quad u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \ldots + a_k u_n$$

- k is the **order** of the sequence
- Its **characteristic polynomial** is

$$p(x) = x^k - a_1 x^{k-1} - a_2 x^{k-2} - \ldots - a_k$$

- The linear recurrence sequence is **simple** if its characteristic polynomial has no repeated roots
- Let $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{C}$ be the characteristic roots. There exist polynomials $p_1(x), p_2(x), \ldots, p_m(x) \in \mathbb{C}[x]$ such that

$$u_n = p_1(n)\lambda_1^n + p_2(n)\lambda_2^n + \ldots + p_m(n)\lambda_m^n$$

  In general $\lambda_1, \ldots, \lambda_k$ and all coefficients of $p_1(x), \ldots, p_m(x)$ are algebraic numbers
- If the linear recurrence sequence is **simple** then the polynomials $p_1(x), \ldots, p_m(x)$ are all **constant**

- Let $\langle u_n \rangle$ be a linear recurrence sequence

Skolem Problem

Does $\exists n$ such that $u_n = 0$?

Decision Problems for Linear Recurrence Sequences

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Is it the case that $\forall n,\ u_n \geq 0$?

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Positivity Problem

Is it the case that $\forall n,\ u_n \geq 0$?

Ultimate Positivity Problem

Does $\exists T$ such that, $\forall n \geq T,\ u_n \geq 0$?

Related Work and Applications

- Theoretical biology
  - Analysis of L-systems
  - Population dynamics
- Software verification
  - Termination of linear programs
- Probabilistic model checking
  - Reachability, invariance, and approximation in Markov chains
  - Stochastic logics
- Quantum computing
  - Threshold problems for quantum automata
- Economics
- Combinatorics
- Discrete linear dynamical systems
- Statistical physics
- . . .

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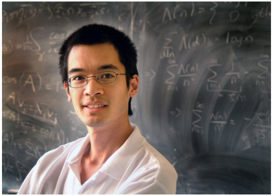

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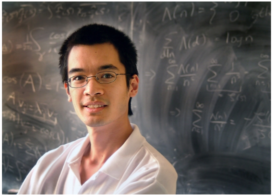

". . . a mathematical embarrassment . . . "

Richard Lipton

The Skolem-Mahler-Lech Theorem

Theorem (Skolem 1934; Mahler 1935, 1956; Lech 1953)

The set of zeros of a linear recurrence sequence is semilinear:

$$\{n : u_n = 0\} = F \cup A_1 \cup \ldots \cup A_\ell$$

where F is finite and each A_i is a full arithmetic progression.

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Theorem (Berstel and Mignotte 1976)

In Skolem-Mahler-Lech, the infinite part (arithmetic progressions $A_1, \ldots, A_\ell$) is fully constructive.

The Skolem Problem at Low Orders

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Critical ingredient is Baker's theorem for linear forms in logarithms, which earned Baker the Fields Medal in 1970.

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Decidability for order 5 was announced in 2005 by four Finnish mathematicians in a technical report (as yet unpublished). Their proof appears to have a serious gap.

The Positivity and Ultimate Positivity Problems

- Positivity and Ultimate Positivity open since at least 1970s

"In our estimation, these will be very difficult problems."

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Decidability of Positivity $\Rightarrow$ decidability of Skolem.

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For order 2, Ultimate Positivity is decidable.

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In *Colloquium Mathematicum* 128:1 (2012), Tangsupphathawat, Punnim, and Laohakosol claimed decidability of Positivity and Ultimate Positivity for order 4 (and noted being stuck for order 5). Unfortunately, their proof contains a major error.

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Positivity is decidable for order 5 or less.

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Theorem

At order 6, for both Positivity and Ultimate Positivity, proof of decidability would entail major breakthroughs in analytic number theory (Diophantine approximation of transcendental numbers).

Some Recent Results (II)

Theorem

For **simple** *linear recurrence sequences of order 9 or less, Positivity is decidable.*

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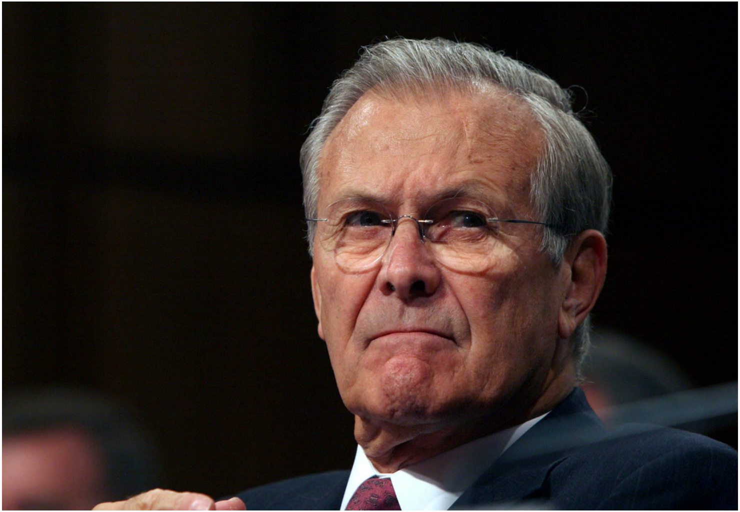
Theorem

*For **simple** linear recurrence sequences, Ultimate Positivity is decidable for **all** orders.*

- *For each fixed order k, complexity is in P (but depends on k).*
- *In the general case, complexity is in PSPACE and co$\exists\mathbb{R}$-hard.*

Known Unknowns



"There are things that we know we don't know. . . "

Donald Rumsfeld

Diophantine Approximation

How well can one approximate a real number x with rationals?

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Moreover, $\frac{1}{\sqrt{5}}$ is the best possible constant that will work for all real numbers x.

Diophantine Approximation

Definition

Let $x \in \mathbb{R}$. The **Lagrange constant** $L_\infty(x)$ is:

$$L_\infty(x) = \inf \left\{ c : \left| x - \frac{p}{q} \right| < \frac{c}{q^2} \text{ has infinitely many solutions} \right\}.$$

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Almost nothing else is known about any specific irrational number!

Hardness

Let $\mathcal{T} = \{\theta \in (0,1) \ : \ e^{2\pi i\theta} \in \mathbb{Q}(i)\} \ \setminus \ \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$

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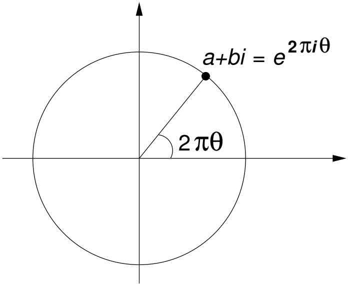

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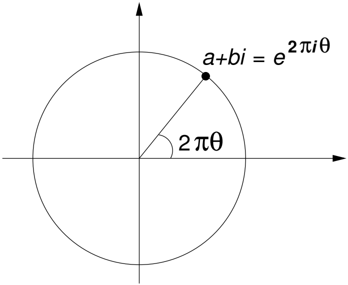

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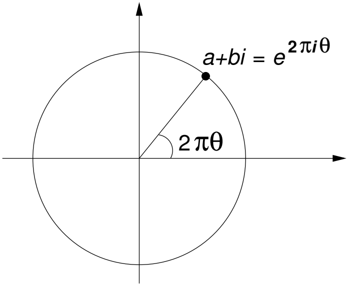

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Theorem

Suppose that Ultimate Positivity is decidable for integer linear recurrence sequences of order 6. Then for any $\theta \in \mathcal{T}$, $L_\infty(\theta)$ is computable.

Positivity of Simple LRS: Algorithm Sketch

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For simple linear recurrence sequences:

- *Ultimate Positivity is decidable for all orders.*
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2. Compute a threshold T such that $\langle u_n \rangle_{n=T}^{\infty}$ is positive.
3. Check individually whether $u_0 \geq 0,\ u_1 \geq 0,\ \ldots,\ u_{T-1} \geq 0$.

Lower Bounds in Diophantine Approximation

Theorem (Dirichlet 1842)

There are infinitely many integers p, q such that $\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$.

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Let $x \in \mathbb{R}$ *be algebraic. Then for any* $\varepsilon > 0$ *there are finitely many integers p, q such that*

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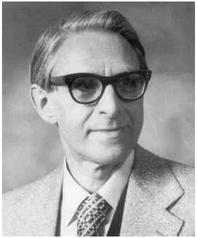

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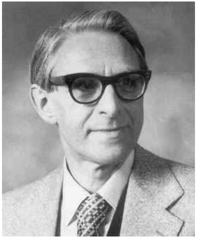

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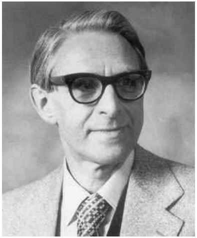

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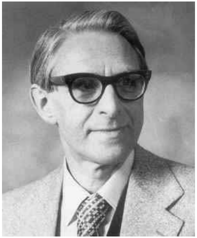

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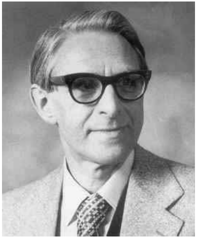

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- $\Rightarrow$ Evertse, van der Poorten, and Schlickewei's **lower bounds on sums of S-units** (1984–1985)

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- How small can the following expression get?

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- Constructive proof requires Baker's Theorem (!)

Lower Bounds on Sums of S-Units: A Simple Example

- How about

$$|\, 3^x \pm 7^y \pm 13^z \,|$$

where $x, y, z \in \mathbb{N}$

Lower Bounds on Sums of S-Units: A Simple Example

- How about

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- For all $\varepsilon > 0$, if x, y, and z are 'large enough', then

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  where $M = \max\{3^x, 7^y, 13^z\}$

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- **No constructive proof is known !**

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We use complex algebraic-integer extensions of such results to study expressions of the form:

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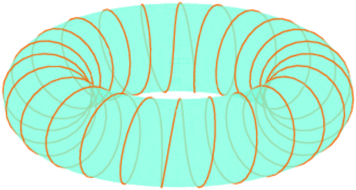

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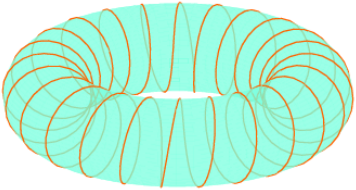
$$f(z_1, z_2, \ldots, z_k) = c_1 z_1 + c_2 z_2 + \ldots + c_k z_k$$



Main Tools and Techniques

- Algebraic and analytic number theory, Diophantine geometry
  - p-adic techniques
  - Baker's theorem on linear forms in logarithms
  - Kronecker's theorem on simultaneous Diophantine approximation
  - Masser's results on multiplicative relationships among algebraic numbers
  - Schmidt's Subspace theorem and Schlickewei's p-adic extension
  - Sums of S-units techniques
  - Gelfond-Schneider theorem
  - Other Diophantine geometry and approximation techniques
- Real algebraic geometry

Decision and Synthesis Problems for Linear Dynamical Systems

- A fresh look at an old area
- Lots of cool problems
- Lots of interesting mathematics
- Many connections to variety of other fields