A Survey of Program Termination: Practical and Theoretical Challenges

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Consider the following order-5 recurrence:

\[ u_{n+5} = -\frac{19}{25} u_{n+4} - \frac{114}{125} u_{n+3} + \frac{114}{125} u_{n+2} + \frac{19}{25} u_{n+1} + u_n \]
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This is simple, with characteristic roots 1, \( \lambda_1, \overline{\lambda_1}, \lambda_2, \overline{\lambda_2} \), where

\[
\lambda_1 = \frac{-3 + 4i}{5} \quad \text{and} \quad \lambda_2 = \frac{-7 + 24i}{25}
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For suitably chosen initial values we have

\[ u_n = \frac{33}{8} + \lambda_1^n + \overline{\lambda_1^n} + 2\lambda_2^n + 2\overline{\lambda_2^n} \]
\( \{ \lambda^n_1 : n \in \mathbb{N} \} \) and \( \{ \lambda^n_2 : n \in \mathbb{N} \} \) are both dense in \( \mathbb{T} \).
Orbits of Characteristic Roots

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- \( \{(\lambda_1^n, \lambda_2^n) : n \in \mathbb{N}\} \) not dense in \( \mathbb{T}^2 \) due to relation \( \lambda_1^2\lambda_2 = 1 \).

Point \((-1, -1)\) does not lie on helix.
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- Point \((-1, -1)\) does not lie on helix.
Critical Point! \( ( -\frac{1}{8} + \frac{\sqrt{63}i}{8}, -\frac{31}{32} + \frac{\sqrt{63}i}{32} ) \)
**Critical Point!**  \((-\frac{1}{8} + \frac{\sqrt{63}i}{8}, -\frac{31}{32} + \frac{\sqrt{63}i}{32})\)

For \((\lambda_1^n, \lambda_2^n)\) near this point,

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u_n := \frac{33}{8} + \lambda_1^n + \overline{\lambda_1^n} + 2\lambda_2^n + 2\overline{\lambda_2^n}
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is close to 0.
Example

- **Critical Point!**  \((-\frac{1}{8} + \frac{\sqrt{63}i}{8}, -\frac{31}{32} + \frac{\sqrt{63}i}{32})\)

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is close to 0.

- \(\langle u_n \rangle\) is ultimately positive—just.
● **Critical Point!** \((-\frac{1}{8} + \frac{\sqrt{63}i}{8}, -\frac{31}{32} + \frac{\sqrt{63}i}{32}\)\)

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● \(\langle u_n \rangle\) is ultimately positive—just.

● But what about \(u_n - \frac{1}{2^n}\)?