Introduction to Permission-Based Program Logics

Part II – Concurrent Programs

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Example: Lock-Coupling List

- There is one lock per node; threads acquire locks in a hand over hand fashion.
- If a node is locked, we can insert a node just after it.
- If two adjacent nodes are locked, we can remove the second.
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Extensions of Separation Logic for Concurrent Programs
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Fig. 1. CSL Family Tree (courtesy of Ilya Sergey)
Extensions of Separation Logic for Concurrent Programs
RGSep Primer
[courtesy of Viktor Vafeiadis]
**Program and Environment**

- **Program**: the current thread being verified.
- **Environment**: all other threads of the system that execute in parallel with the thread being verified.
- **Interference**: The program interferes with the environment by modifying the shared state.

Conversely, the environment interferes with the program by modifying the shared state.
Local & Shared State

- The total state is logically divided into two components:
  - **Shared**: accessible by all threads via synchronization
  - **Local**: accessible only by one thread, its owner

State of the lock-coupling list just before inserting a new node.
The node to be added is local because other threads cannot yet access it.
Program Specifications

• The specification of a program consists of two assertions (precondition & postcondition), and two sets of actions:

• **Rely:** Describes the interference that the program can tolerate from the environment; i.e. specifies how the environment can change the shared state.

• **Guarantee:** Describes the interference that the program imposes on its environment; i.e. specifies how the program can change the shared state.
Rely/Guarantee Actions

Actions describe minimal atomic changes to the shared state.

Lock node

Unlock node

An action allows any part of the *shared state* that satisfies the LHS to be changed to a part satisfying the RHS, but the rest of the shared state must not be changed.
Rely/Guarantee Actions

Actions can adjust the boundary between local state and stared state.
This is also known as *transfer of ownership*.

Add node

Delete node
Rely/Guarantee Actions

Actions can adjust the boundary between local state and shared state. This is also known as *transfer of ownership*.
Rely/Guarantee Actions: Lock Coupling List

Add node
Rely/Guarantee Actions: Lock Coupling List
Rely/Guarantee Actions: Lock Coupling List

shared: 2 → 3 → 5 → 7 → 6 → 8 → 9

local

Add node:
Rely/Guarantee Actions: Lock Coupling List

shared 2 3 5 7 8 9

local

Add node
Rely/Guarantee Actions: Lock Coupling List

shared 2 → 3 → 5 → 7 → 8 → 9

local

Lock node
Rely/Guarantee Actions: Lock Coupling List

shared

local

Lock node
Rely/Guarantee Actions:
Lock Coupling List

shared 2 → 3 → 5 → 7 → 8 → 9

local

Lock node → Lock node
Rely/Guarantee Actions: Lock Coupling List

shared 2 → 3 → 5 → 7 → 8 → 9

local

Lock node
Rely/Guarantee Actions: Lock Coupling List

shared

local

Delete node
Rely/Guarantee Actions: Lock Coupling List

shared 2 3 5 6 7 8 9

local

Delete node

arrow
Rely/Guarantee Actions: Lock Coupling List

Delete node
Rely/Guarantee Actions:
Lock Coupling List

shared

local

Delete node
Assertion Syntax

- Separation Logic
  \[ P, Q ::= e = e \mid e \neq e \mid e \mapsto (f : e) \mid P \ast Q \mid \ldots \]

- Extended Logic
  \[ p, q ::= P \mid [P] \mid p \ast q \mid \ldots \]

local \hspace{2cm} shared
Assertion Semantics

• $l, s \models P \iff l \models_{SL} P$

• $l, s \models \mathbf{P} \iff s \models_{SL} P$ and $l = \emptyset$

• $l, s \models p \cdot q \iff$ exists $l_1, l_2$:
  
  $l = l_1 \bullet l_2$ and $l_1, s \models p$ and $l_2, s \models q$
Assertion Semantics

• $l, s \models P \iff l \models_{SL} P$
• $l, s \models P \iff s \models_{SL} P$ and $l = \emptyset$
• $l, s \models p \ast q \iff \exists l_1, l_2: \begin{align*}
l &= l_1 \bullet l_2 \text{ and } l_1, s \models p \text{ and } l_2, s \models q
\end{align*}$

split local state
Assertion Semantics

• \( l, s \models P \iff l \models_{SL} P \)

• \( l, s \models \boxed{P} \iff s \models_{SL} P \) and \( l = \emptyset \)

• \( l, s \models p \ast q \iff \) exists \( l_1, l_2 : \)
  \[
  l = l_1 \bullet l_2 \text{ and } l_1, s \models p \text{ and } l_2, s \models q
  \]

share global state
Assertions: Lock Coupling List

Unlocked node $x$ holding value $v$ and pointing to $y$

$$x \mapsto (0, v, y)$$

Node $x$ holding value $v$ and pointing to $y$, locked by thread $T$

$$x \mapsto (T, v, y)$$

List segment from $x$ to $y$ of possibly locked nodes

$$lseg(x, y)$$
Rely/Guarantee Actions: Lock Coupling List

\[ x \mapsto (0, v, y) \rightarrow x \mapsto (T, v, y) \]

\[ x \mapsto (T, v, y) \rightarrow x \mapsto (0, v, y) \]

\[ x \mapsto (T, v, y) \rightarrow x \mapsto (0, v, y) \]

\[ z \mapsto (0, w, y) \]

\[ x \mapsto (T, v, z) \rightarrow * \]

\[ z \mapsto (T, w, y) \]
Programs: Syntax

• Basic commands c:
  – noop: skip
  – guard: assume(b)
  – heap write: [x] := y
  – heap read: x := [y]
  – allocation: x := new()
  – deallocation: free(x)
  – ...

• Commands C ∈ Com:
  – basic commands: c
  – seq. composition: C₁; C₂
  – nondet. choice: C₁ + C₂
  – looping: C*
  – atomic com.: atomic C
  – par. composition: C₁ | C₂
Rely/Guarantee Judgements

\[ \vdash C \text{ sat } (p, R, G, q) \]

(precondition, rely, guarantee, postcondition)
Parallel Composition Rule

\[ \vdash C_1 \text{ sat} (p_1, R \cup G_2, G_1, q_1) \]
\[ \vdash C_2 \text{ sat} (p_2, R \cup G_1, G_2, q_2) \]
\[ \vdash (C_1 \mid C_2) \text{ sat} (p_1 \ast p_2, R, G_1 \cup G_2, q_1 \ast q_2) \]
Stability

• An assertion is *stable* iff it is preserved under interference by other threads.

• Example:
Stability

• An assertion is *stable* iff it is preserved under interference by other threads.

• Example:
Stability

• An assertion is **stable** iff it is preserved under interference by other threads.

• Example:

```
2 3 5 7 8 9
```

```html
<Diagram showing lock and not stable>
```

not stable!
Stability

• An assertion is *stable* iff it is preserved under interference by other threads.

• Example:

![Diagram of a lock and unlock sequence showing stability](image)
Stability

• An assertion is *stable* iff it is preserved under interference by other threads.

• Example:

![Diagram showing stability example]
Stability

• An assertion is *stable* iff it is preserved under interference by other threads.

• Example:
Stability

• An assertion is \textit{stable} iff it is preserved under interference by other threads.

• Example:

\begin{itemize}
  \item Delete
  \begin{itemize}
    \item \textbf{B}
    \item \textbf{B}
  \end{itemize}
  \begin{itemize}
    \item \textbf{5}
    \item \textbf{7}
  \end{itemize}
  \begin{itemize}
    \item \textbf{B}
    \item \textbf{B}
  \end{itemize}
\end{itemize}

\textbf{stable!}
Stability (Formally)

\[ S \text{ stable under } P \rightarrow Q \iff (P \ominus S) * Q \models S \]

where \( P \ominus S \) := \( \neg (\neg P \ominus S) \)
Atomic Commands

\[ \vdash \{ P \} C \{ Q \} \]

\[ \vdash (\text{atomic } C) \text{ sat } (P, \emptyset, \emptyset, Q) \]
Atomic Commands

\[
\vdash \{ P \} C \{ Q \}
\]

\[
\vdash (\text{atomic } C) \text{ sat}(P, \emptyset, \emptyset, Q)
\]

reduction to sequential SL

only local state
Atomic Commands

\[ \vdash \{ P \} C \{ Q \} \]
\[ \vdash (\text{atomic } C) \text{ sat } (P, \emptyset, \emptyset, Q) \]
\[ p, q \text{ stable under } R \]
\[ \vdash (\text{atomic } C) \text{ sat } (p, \emptyset, G, q) \]
\[ \vdash (\text{atomic } C) \text{ sat } (p, R, G, q) \]
Atomic Commands

\[ P_2, Q_2 \text{ precise} \quad P_2 \rightarrow Q_2 \in G \]

\[ \vdash (\text{atomic } C) \text{ sat } (P_1 \ast P_2, \emptyset, \emptyset, Q_1 \ast Q_2) \]

\[ \vdash (\text{atomic } C) \text{ sat } (P_1 \ast \boxed{P_2 \ast F}, \emptyset, G, Q_1 \ast \boxed{Q_2 \ast F}) \]
Atomic Commands

$P_2, Q_2$ precise  \[ P_2 \rightarrow Q_2 \in G \]

\[ \vdash (\text{atomic } C) \text{ sat } (P_1 * P_2, \emptyset, \emptyset, Q_1 * Q_2) \]

\[ \vdash (\text{atomic } C) \text{ sat } (P_1 * P_2 * F, \emptyset, G, Q_1 * Q_2 * F) \]
Atomic Commands

\[ P_2, Q_2 \text{ precise} \quad P_2 \rightarrow Q_2 \in G \]

\[ \vdash (\text{atomic } C) \text{ sat}(P_1 * P_2, \emptyset, \emptyset, Q_1 * Q_2) \]

\[ \vdash (\text{atomic } C) \text{ sat}(P_1 * \boxed{P_2 * F}, \emptyset, G, Q_1 * \boxed{Q_2 * F}) \]
Atomic Commands

$P_2, Q_2$ precise $\quad P_2 \rightarrow Q_2 \in G$

$\vdash (\text{atomic } C) \text{ sat } (P_1 \ast P_2, \emptyset, \emptyset, Q_1 \ast Q_2)$

$\vdash (\text{atomic } C) \text{ sat } (P_1 \ast \boxed{P_2 \ast F}, \emptyset, G, Q_1 \ast \boxed{Q_2 \ast F})$
Atomic Commands

$P_2, Q_2$ precise \hfill $P_2 \rightarrow Q_2 \in G$

$\vdash (\text{atomic } C) \text{ sat } (P_1 \ast P_2, \emptyset, \emptyset, Q_1 \ast Q_2)$

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Atomic Commands

\[ P_2, Q_2 \text{ precise} \quad P_2 \rightarrow Q_2 \in G \]

\[ \vdash \text{(atomic C) sat} \left( P_1 \ast P_2, \emptyset, \emptyset, Q_1 \ast Q_2 \right) \]

\[ \vdash \text{(atomic C) sat} \left( P_1 \ast P_2 \ast \mathbf{F}, \emptyset, G, Q_1 \ast Q_2 \ast \mathbf{F} \right) \]

Diagram:
- Shared: 2 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 8 \rightarrow 9
- Local: \quad P_2 \rightarrow Q_2

Q_2

Q_2
Atomic Commands

$P_2, Q_2$ precise \hspace{1cm} $P_2 \rightarrow Q_2 \in G$

$\vdash (\text{atomic } C) \text{ sat } (P_1 \ast P_2, \emptyset, \emptyset, Q_1 \ast Q_2)$

$\vdash (\text{atomic } C) \text{ sat } (P_1 \ast \framebox{P_2 \ast F}, \emptyset, G, Q_1 \ast \framebox{Q_2 \ast F})$
Atomic Commands

$P_2, Q_2$ precise  \[ P_2 \rightarrow Q_2 \in G \]

\[ \vdash (\text{atomic } C) \text{sat} (P_1 \ast P_2, \emptyset, \emptyset, Q_1 \ast Q_2) \]

\[ \vdash (\text{atomic } C) \text{sat} (P_1 \ast [\underline{P_2 \ast F}], \emptyset, G, Q_1 \ast [\underline{Q_2 \ast F}]) \]

Q_1 = \text{emp}

**Diagram:**
- Shared:
  - Nodes: 2, 3, 5, 7, 8, 9
- Local:
  - Nodes: P_2, Q_2

Nodes are connected with arrows indicating the flow or transition between states.
Challenge: Harris' Non-blocking List
Challenge: Harris' Non-blocking List
Challenge: Harris' Non-blocking List

head → -∞ → 3 → 6 → 8 → ∞
Challenge: Harris' Non-blocking List
Challenge: Harris' Non-blocking List
Challenge: Harris' Non-blocking List

(head) → (-∞) → 3 → 6 → 8 → (∞)
Challenge: Harris' Non-blocking List

head

-∞ → 3 → 6 → 8 → ∞

free

1 → 7 → 2
Challenge: Harris' Non-blocking List
Challenge: Harris' Non-blocking List

head

-∞ → 3 → 6 → ∞

free

1 → 7 → 2 → 8

ZZ
Challenge: Harris' Non-blocking List
Challenge: Harris' Non-blocking List
Challenge: Harris' Non-blocking List
Challenge: Harris' Non-blocking List
Flow Interfaces

joint work with Siddharth Krishna and Dennis Shasha
Goal

• Data structure abstractions that
  – can handle unbounded sharing and overlays
  – treat structural and data constraints uniformly
  – do not encode specific traversal strategies
  – provide data-structure-agnostic composition and decomposition rules
  – remain within general theory of separation logic

⇒ Flow Interfaces
High-Level Idea

• Express all data structure invariants in terms of a local condition, satisfied by each node.
  – Local condition may depend on a quantity of the graph that is calculated inductively over the entire graph (the flow).

• Introduce a notion of graph composition that preserves local invariants of global flows.

• Introduce a generic *good graph* predicate that abstracts a heap region satisfying the local flow condition (the *flow interface*).
Can we express the property that root points to a tree as a local condition of each node in the graph?
Can we express the property that root points to a tree as a local condition of each node in the graph?
Can we express the property that \texttt{root} points to a tree as a local condition of each node in the graph?

\[
\forall n \in \mathbb{N}. \text{pc}(\texttt{root}, n) \leq 1
\]

"G contains a tree rooted at \texttt{root}"
Flows

Step 1: Defining the Flow Graph

Label each edge in the graph with an element from some flow domain \((D, \sqsubseteq, +, \cdot, 0, 1)\)
Flows

Step 1: Defining the Flow Graph

Requirements of flow domain:
• \((D, +, \cdot, 0, 1)\) is a semiring
• \((D, \sqsubseteq)\) is \(\omega\)-cpo with smallest element 0
• + and \(\cdot\) are continuous

Path counting flow domain:
\((\mathbb{N} \cup \{\infty\}, \leq, +, \cdot, 0, 1)\)

Label each edge in the graph with an element from some flow domain \((D, \sqsubseteq, +, \cdot, 0, 1)\)
Flows

Step 1: Defining the Flow Graph

Flow graph $G = (N, e)$
- $N$ finite set of nodes
- $e: N \times N \rightarrow D$

Label each edge in the graph with an element from some
flow domain $(D, \sqsubseteq, +, \cdot, 0, 1)$
Flows

Step 1: Defining the Flow Graph

Flow graph $G = (N, e)$
- $N$ finite set of nodes
- $e : N \times N \to D$

Label each edge in the graph with an element from some *flow domain* $(D, \sqsubseteq, +, \cdot, 0, 1)$
Flows

Step 2: Define the Inflow

Label each node using an *inflow* $in: N \rightarrow D$

$$in_{\text{root}}(n) = \begin{cases} 1, & n = \text{root} \\ 0, & n \neq \text{root} \end{cases}$$
Flows

Step 3: Calculate the flow

Flow graph $G = (N, e)$

$$\text{flow}(in, G) : N \rightarrow D$$

$$\text{flow}(in, G) = \text{lfp} \left( \lambda C. \lambda n \in N. \text{in}(n) + \sum_{n' \in N} C(n') \cdot e(n', n) \right)$$
Flows

Step 3: Calculate the flow

Flow graph $G = (N, e)$

flow$((in, G) : N \rightarrow D$

flow$((in, G) = \text{lfp} \left( \lambda C. \lambda n \in N. \text{in}(n) + \sum_{n' \in N} C(n') \cdot e(n', n) \right)$
Flows
Step 3: Calculate the flow

Flow graph $G = (N, e)$

$\forall n \in N. \text{flow}(in_{\text{root}}, G)(n) \leq 1$

"G contains a tree rooted at root"

flow($in, G$) : $N \rightarrow D$

$\text{flow}(in, G) = \lambda C. \lambda n \in N. \text{in}(n) + \sum_{n' \in N} C(n') \cdot e(n', n)$
Data Constraints

\[
\text{predicate } \text{tree}(t: \text{Node}, \ C: \text{Set}\langle\text{Int}\rangle) \{ \\
\quad t == \text{null} \land \text{emp} \land \ C == \emptyset \lor \\
\quad \exists \ v, \ x, \ y, \ Cx, \ Cy :: \\
\quad \quad t \mapsto (d:v, r:x, l:y) \ast \text{tree}(x, Cx) \ast \text{tree}(y, Cy) \land \\
\quad \quad C == \{v\} \cup Cx \cup Cy \land v > Cx \land v < Cy \\
\}
\]
Data Invariants

**predicate** `tree(t: Node, C: Set<Int>)` {
  `t == null ∧ emp ∧ C = ∅ ∨`
  `∃ v, x, y, Cx, Cy :: t = (d:v, r:x, l:y) * tree(x, Cx) * tree(y, Cy) ∧ C = {v} ∪ Cx ∪ Cy ∧ v > Cx ∧ v < Cy`
}

```
implies Cx ∩ Cy = ∅
```
Data Invariants

**predicate** tree(t: Node, C: Set<Int>) {
    t == null ∧ emp ∧ C = ∅ 
    ∃ v, x, y, Cx, Cy ::
        t → (d:v, r:x, l:y) * tree(x, Cx) * tree(y, Cy) ∧
        C = {v} ∪ Cx ∪ Cy ∧ v > Cx ∧ v < Cy
}

Data invariant piggybacks on inductive definition of the tree.
⇒ hard to entangle data invariants from data structure specifics.
Inset Flows

KS: the set of all search keys
e.g. $KS = \mathbb{Z}$

Inset flow domain:
$(2^KS, \subseteq, \cup, \cap, \emptyset, KS)$

Label each edge with the set of keys that follow that edge in a search (edgeset).
Inset Flows

KS: the set of all search keys
e.g. KS = \mathbb{Z}

Inset flow domain:
\((2^{KS}, \subseteq, \cup, \cap, \emptyset, KS)\)

Label each edge with the set of keys that follow that edge in a search (edgeset).
Inset Flows

KS: the set of all search keys e.g. KS = \( \mathbb{Z} \)

Inset flow domain: 
\((2^K_S, \subseteq, \cup, \cap, \emptyset, K_S)\)

Set inflow in of root to KS and to \( \emptyset \) for all other nodes.
Inset Flows

flow(in, G)(n) is the *inset* of node n, i.e., the set of keys k such that a search for k will traverse node n.

\[ I_1 = \{ k \mid 3 < k \} \]
\[ I_2 = \{ k \mid 3 < k < 6 \} \]
\[ I_3 = \{ k \mid 8 < k \} \]
From Insets to Keysets

\[
\text{outset}(G)(n) = \bigcup_{n \in N} e(n, n')
\]

\[
\text{keyset}(\text{in}, G)(n) = \text{inset}(\text{in}, G)(n) \setminus \text{outset}(G)(n)
\]

keyset(in, G)(n) is the set of keys that could be in n.
Verifying Concurrent Search Data Structures

• Local data structure invariants
  – edgesets are disjoint for each $n$:
    \[ \{e(n,n')\}_{n' \in N} \text{ are disjoint} \]
  – keyset of each $n$ covers $n$'s contents:
    \[ C(G)(n) \subseteq \text{keyset}(\text{in}, G)(n) \]

• Observation: disjoint inflows imply disjoint keysets
  – If \( \{\text{in}(n)\}_{n \in N} \) are disjoint (e.g. $G$ has a single root)
  – then \( \{\text{keyset}(\text{in}, G)(n)\}_{n \in N} \) are disjoint

\[ \Rightarrow \] Can be used to prove linearizability of concurrent search data structures in a data-structure-agnostic fashion

[Shasha and Goodman, 1988]
Compositional Reasoning

Can we reason compositionally about flows and flow graphs à la SL?
Flow Graph Composition

- Standard SL Composition (disjoint union) is too weak:

```
\[ \begin{array}{c}
x \\ 1 \\ \downarrow \\ y \\ 1 \\
\end{array} \ast \begin{array}{c}
x \\ 1 \\ \downarrow \\ y \\ 1 \\
\end{array} = \begin{array}{c}
x \\ 1 \\ \downarrow \\ y \\ 1 \\
\end{array} \\
\text{a tree} \ast \text{a tree} = \text{not a tree}
```

- A tree: A connected graph with no cycles.
- Not a tree: A graph with one or more cycles.
Flow Interface Graph

\((in, G)\) is a *flow interface graph* iff

- \(G = (N, N_o, \lambda, e)\) is a partial graph with
  - \(N\) the set of internal nodes of the graph
  - \(N_o\) the set of external nodes of the graph
  - \(\lambda: N \rightarrow A\) a node labeling function
  - \(e: N \times (N \cup N_o) \rightarrow D\) is an edge function
  - \(in: N \rightarrow D\) is an inflow

Inflow \(in\) specifies *rely* of \(G\) on its context.
Flow Interface Graph Composition

\((in, G)\)
Flow Interface Graph Composition

\((\mathit{in}, \mathcal{G}) = (\mathit{in}_1, \mathcal{G}_1) \bullet (\mathit{in}_2, \mathcal{G}_2)\)

\(\mathit{in}_1 = ?, \mathit{in}_2 = ?\)
Flow Interface Graph Composition

\[(\text{in}, G) = (\text{in}_1, G_1) \bullet (\text{in}_2, G_2)\]
Flow Interface Graph Composition

\[(\text{in}, G) = (\text{in}_1, G_1) \bullet (\text{in}_2, G_2)\]
Flow Interface Graph Composition

\[(in, G) = (in_{1}, G_{1}) \bullet (in_{2}, G_{2})\]
Flow Interface Graph Composition

• $H_1 \bullet H_2$ is
  – commutative: $H_1 \bullet H_2 = H_2 \bullet H_1$
  – associative: $(H_1 \bullet H_2) \bullet H_3 = H_1 \bullet (H_2 \bullet H_3)$
  – cancelative: $H \bullet H_1 = H \bullet H_2 \Rightarrow H_1 = H_2$

⇒ Flow interface graphs form a separation algebra.
⇒ We can use them to give semantics to SL assertions.

• How do we abstract flow interface graphs?
Flow Map of a Flow Interface Graph

\[ fm(G)(n, n_o) = \sum \{ \text{pathproduct}(p) \mid p \text{ path from } n \text{ to } n_o \text{ in } G \} \]

\[ \text{flow}(in, G)(n_o) = \sum \{ \text{in}(n) \cdot fm(G)(n, n_o) \mid n \in G \} \]
Flow Map of a Flow Interface Graph

\[
\text{fm}(G)(n, n_\circ) = \sum \{ \text{pathproduct}(p) \mid p \text{ path from } n \text{ to } n_\circ \text{ in } G \}
\]

\[
\text{flow}(in, G)(n_\circ) = \sum \{ \text{in}(n) \cdot \text{fm}(G)(n, n_\circ) \mid n \in G \}
\]
Flow Map of a Flow Interface Graph

\[ fm(G)(n, n_o) = \sum \{ \text{pathproduct}(p) \mid p \text{ path from } n \text{ to } n_o \text{ in } G \} \]

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Flow Map of a Flow Interface Graph

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fm(G)(n, n_o) = \sum \{ pathproduct(p) \mid p \text{ path from } n \text{ to } n_o \text{ in } G \}
\]

\[
\text{flow}(in, G)(n_o) = \sum \{ in(n) \cdot fm(G)(n, n_o) \mid n \in G \}
\]
Flow Map: Example

Flow map abstracts from internal structure of the graph
Flow Map: Example

Flow map abstracts from internal structure of the graph
Flow Map: Example

Flow map abstracts from internal structure of the graph
Flow Interfaces

• $I = (in, f)$ is a flow interface if
  – $in: \mathbb{N} \rightarrow D$ is an inflow
  – $f: \mathbb{N} \times \mathbb{N}_o \rightarrow D$ is a flow map

• $\llbracket (in, f) \rrbracket_{good}$ denotes all flow interface graphs $(in, G)$ s.t.
  – $fm(G) = f$
  – for all $n \in \mathbb{N}$ $good(in(n), G|_n)$ holds

• where $good$ is some good node condition
  – e.g. $good(i, _) = i \leq 1$
Flow Interfaces with Node Abstraction

• $I = (in, \alpha, f)$ is a flow interface if
  – $in: N \to D$ is an inflow
  – $f: N \times N_o \to D$ is a flow map
  – $\alpha \in A$ is a node label

• $\llbracket (in, \alpha, f) \rrbracket_{good}$ denotes all flow interface graphs $(in, G)$ s.t.
  – $fm(G) = f$
  – $\alpha = \bigsqcup \{ \lambda_G(n) \mid n \in N \}$
  – for all $n \in N$ $good(in(n), G | n)$ holds

• where $good$ is some good node condition
  – e.g. $good(i, _) = i \leq 1$
Flow Interface Composition

Composition of flow interface graphs can be lifted to flow interfaces:

- \( I \in I_1 \oplus I_2 \) iff \( \exists H, H_1, H_2 \) such that
  - \( H \in \llbracket I \rrbracket, H_1 \in \llbracket I_1 \rrbracket \), and \( H_2 \in \llbracket I_2 \rrbracket \)
  - \( H = H_1 \bullet H_2 \)

Some nice properties of \( \oplus \):

- \( \oplus \) is associative and commutative
- \( \llbracket I_1 \rrbracket \bullet \llbracket I_2 \rrbracket \subseteq \llbracket I_1 \oplus I_2 \rrbracket \)
- if \( I \in I_1 \oplus I_2 \), then for all \( H_1 \in \llbracket I_1 \rrbracket \), \( H_2 \in \llbracket I_2 \rrbracket \), \( H_1 \bullet H_2 \) defined
- ...
Separation Logic with Flow Interfaces

- Good graph predicate $\text{Gr}_\gamma(I)$
  - $\gamma$: SL predicate that defines good node condition and abstraction of heap onto nodes of flow graph
  - $I$: flow interface term

- Good node predicate $\text{N}_\gamma(x, I)$
  - like $\text{Gr}$ but denotes a single node
  - definable in terms of $\text{Gr}$

- Dirty region predicate $[P]_{\gamma,I}$
  - $P$: SL predicate
  - denotes heap region that is expected to satisfy interface $I$ but may currently not
Graph Predicate: Linked List

- Abstraction of linked list node

\[ \gamma(x, \text{in}, C, f) = \exists k, y. \ x \mapsto (\text{data}: k, \text{next}: y) \land \]
\[ C = \{k\} \land k \in \text{in} \land \]
\[ f = \text{ITE}(y = \text{null}, \epsilon, \{ (x,y) \mapsto \{k', k' > k\} \}) \]

- Invariant

\[ \exists I :: \text{Gr}_\gamma(I) \land I^{\text{in}} = \{\text{root} \mapsto \text{KS}\}.0 \land I^f = \epsilon \]
Graph Predicate: Binary Search Tree

• Abstraction of BST node

\[ \gamma(x, \text{in}, C, f) = \exists k, y, z. x \mapsto (\text{data}: k, \text{left}: y, \text{right}: z) \land \]
\[ C = \{k\} \land k \in \text{in} \land \]
\[ f = \text{ITE}(y = \text{null}, \varepsilon, \{(x, y) \mapsto \{k'. k' < k\}\}). \]
\[ \text{ITE}(z = \text{null}, \varepsilon, \{(x, z) \mapsto \{k'. k' > k\}\}) \]

• Invariant

\[ \exists I :: \text{Gr}_{\gamma}(I) \land I^{in} = \{\text{root} \mapsto \text{KS}\}.0 \land I^{f} = \varepsilon \]
Graph Predicate: Binary Search Tree

Need tree invariant?

• Abstraction of BST node

\[ \gamma(x, \text{in}, C, f) = \exists k, y, z. \ x \mapsto (\text{data: } k, \ \text{left: } y, \ \text{right: } z) \land \]
\[
C = \{k\} \land k \in \text{in} \land \\
f = \text{ITE}(y = \text{null}, \epsilon, \{ (x,y) \mapsto \{k'. k' < k\} \} ). \\
\text{ITE}(z = \text{null}, \epsilon, \{ (x,z) \mapsto \{k'. k' > k\} \})
\]

• Invariant

\[ \exists I :: \text{Gr}_\gamma(I) \land I^{in} = \{\text{root} \mapsto \text{KS}\}.0 \land I^f = \epsilon \]
Graph Predicate: Binary Search Tree

Need tree invariant?
No problem!

• Abstraction of BST node

\[ \gamma(x, (in, pc), C, f) = \exists k, y, z. x \mapsto (data: k, left: y, right: z) \land C = \{k\} \land k \in in \land pc = 1 \land f = ITE(y = null, \epsilon, \{ (x,y) \mapsto (\{k'. k' < k\}, 1) \} \land ITE(z = null, \epsilon, \{ (x,z) \mapsto (\{k'. k' > k\}, 1) \} \}

• Invariant

\[ \exists I :: Gr_\gamma(I) \land I^{in} = \{ \text{root} \mapsto (KS, 1) \}.0 \land I^f = \epsilon \]
Data-Structure-Agnostic Proof Rules

Decomposition
\[ \text{Gr}(I) \land x \in I^{\text{in}} \]
\[ N(x, I_1) \ast \text{Gr}(I_2) \land I \in I_1 \oplus I_2 \]

Abstraction
\[ \text{Gr}(I_1) \ast \text{Gr}(I_2) \land I \in I_1 \oplus I_2 \]
\[ \text{Gr}(I) \land I \in I_1 \oplus I_2 \]

Replacement
\[ I \in I_1 \oplus I_2 \land I_1 \preceq J_1 \]
\[ J \in J_1 \oplus I_2 \land I \preceq J \]
Generic R/G Actions

- **Lock node** $N(x, (\text{in}, 0, f)) \rightarrow N(x, (\text{in}, T, f))$
- **Unlock node** $N(x, (\text{in}, T, f)) \rightarrow N(x, (\text{in}, 0, f))$
- **Dirty** $[true]_I \land I^\alpha = t \rightarrow [true]_I$
- **Sync** $[true]_I \land I^\alpha = t \rightarrow \text{Gr}(I') \land I \preceq I'$
Conclusion

• Radically new approach for building compositional abstractions of data structures.
• Fits in existing (concurrent) separation logics.
• Enables simple correctness proofs of concurrent data structure algorithms
• Proofs abstract from the details of the specific data structure implementation.