3.5 Normal Forms and Skolemization (Traditional)

Study of normal forms motivated by

- reduction of logical concepts,
- efficient data structures for theorem proving.

The main problem in first-order logic is the treatment of quantifiers. The subsequent normal form transformations are intended to eliminate many of them.

Prenex Normal Form

Prenex formulas have the form

$$Q_1 x_1 \ldots Q_n x_n \ F,$$

where F is quantifier-free and $Q_i \in \{\forall, \exists\}$; we call $Q_1 x_1 \ldots Q_n x_n$ the *quantifier prefix* and F the *matrix* of the formula.

Computing prenex normal form by the rewrite relation $\Rightarrow_P$:

$$\begin{array}{rll} (F \leftrightarrow G) & \Rightarrow_P & (F \rightarrow G) \land (G \rightarrow F) \\ \neg QxF & \Rightarrow_P & \overline{Q}x\neg F \qquad\qquad (\neg Q) \\ ((QxF)\ \rho\ G) & \Rightarrow_P & Qy(F[y/x]\ \rho\ G),\ y \text{ fresh},\ \rho \in \{\land, \lor\} \\ ((QxF) \rightarrow G) & \Rightarrow_P & \overline{Q}y(F[y/x] \rightarrow G),\ y \text{ fresh} \\ (F\ \rho\ (QxG)) & \Rightarrow_P & Qy(F\ \rho\ G[y/x]),\ y \text{ fresh},\ \rho \in \{\land, \lor, \rightarrow\} \end{array}$$

Here $\overline{Q}$ denotes the quantifier *dual* to Q, i. e., $\overline{\forall} = \exists$ and $\overline{\exists} = \forall$.

Skolemization

Intuition: replacement of $\exists y$ by a concrete choice function computing y from all the arguments y depends on.

Transformation $\Rightarrow_S$ (to be applied outermost, *not* in subformulas):

$$\forall x_1, \ldots, x_n \exists y F \ \Rightarrow_S \ \forall x_1, \ldots, x_n F[f(x_1, \ldots, x_n)/y]$$

where f, where $\mathsf{arity}(f) = n$, is a new function symbol (*Skolem function*).

Together: $F \overset{*}{\Rightarrow}_P \underbrace{G}_{\text{prenex}} \overset{*}{\Rightarrow}_S \underbrace{H}_{\text{prenex, no } \exists}$

Theorem 3.9 *Let F, G, and H as defined above and closed. Then* *(i)* F *and* G *are equivalent.*

(ii) $H \models G$ *but the converse is not true in general.*

(iii) G *satisfiable (w. r. t.* Σ-*Alg)* $\Leftrightarrow$ H *satisfiable (w. r. t.* Σ'-*Alg) where* $\Sigma' = (\Omega \cup SKF, \Pi)$, *if* $\Sigma = (\Omega, \Pi)$.

Clausal Normal Form (Conjunctive Normal Form)

$$
\begin{array}{rcl}
(F \leftrightarrow G) & \Rightarrow_K & (F \rightarrow G) \wedge (G \rightarrow F) \\
(F \rightarrow G) & \Rightarrow_K & (\neg F \vee G) \\
\neg(F \vee G) & \Rightarrow_K & (\neg F \wedge \neg G) \\
\neg(F \wedge G) & \Rightarrow_K & (\neg F \vee \neg G) \\
\neg\neg F & \Rightarrow_K & F \\
(F \wedge G) \vee H & \Rightarrow_K & (F \vee H) \wedge (G \vee H) \\
(F \wedge \top) & \Rightarrow_K & F \\
(F \wedge \bot) & \Rightarrow_K & \bot \\
(F \vee \top) & \Rightarrow_K & \top \\
(F \vee \bot) & \Rightarrow_K & F
\end{array}
$$

These rules are to be applied modulo associativity and commutativity of $\wedge$ and $\vee$. The first five rules, plus the rule $(\neg Q)$, compute the *negation normal form* (NNF) of a formula.

The Complete Picture

$$
\begin{array}{rll}
F & \Rightarrow^*_P \; Q_1 y_1 \ldots Q_n y_n \; G & (G \text{ quantifier-free}) \\
 & \Rightarrow^*_S \; \forall x_1, \ldots, x_m \; H & (m \leq n,\ H \text{ quantifier-free}) \\
 & \Rightarrow^*_K \; \underbrace{\underbrace{\forall x_1, \ldots, x_m}_{\text{leave out}} \; \bigwedge_{i=1}^{k} \; \underbrace{\bigvee_{j=1}^{n_i} L_{ij}}_{\text{clauses } C_i}}_{F'} &
\end{array}
$$

$N = \{C_1, \ldots, C_k\}$ is called the *clausal (normal) form* (CNF) of F.
Note: the variables in the clauses are implicitly universally quantified.

Theorem 3.10 *Let* F *be closed. Then* $F' \models F$. *(The converse is not true in general.)*

Theorem 3.11 *Let* F *be closed. Then* F *is satisfiable iff* F' *is satisfiable iff* N *is satisfiable*

Optimization

Here is lots of room for optimization since we only can preserve satisfiability anyway:

- size of the CNF exponential when done naively;
  but see the transformations we introduced for propositional logic
- want to preserve the original formula structure;
- want small arity of Skolem functions (follows)

3.6 Getting small Skolem Functions

- produce a negation normal form (NNF)
- apply miniscoping
- rename all variables
- skolemize

Negation Normal Form (NNF)

Apply the rewrite relation $\Rightarrow_{NNF}$, F is the overall formula:

$$\begin{array}{rcl}
G \leftrightarrow H & \Rightarrow_{NNF} & (G \rightarrow H) \wedge (H \rightarrow G) \\
& & \text{if } F/p = G \leftrightarrow H \text{ and } F/p \text{ has positive polarity} \\
G \leftrightarrow H & \Rightarrow_{NNF} & (G \wedge H) \vee (\neg H \wedge \neg G) \\
& & \text{if } F/p = G \leftrightarrow H \text{ and } F/p \text{ has negative polarity} \\
\neg Qx\, G & \Rightarrow_{NNF} & \overline{Q}x \neg G \\
\neg(G \vee H) & \Rightarrow_{NNF} & \neg G \wedge \neg H \\
\neg(G \wedge H) & \Rightarrow_{NNF} & \neg G \vee \neg H \\
G \rightarrow H & \Rightarrow_{NNF} & \neg G \vee H \\
\neg\neg G & \Rightarrow_{NNF} & G
\end{array}$$

Miniscoping

Apply the rewrite relation $\Rightarrow_{MS}$. For the below rules we assume that x occurs freely in G, H, but x does not occur freely in F:

$$\begin{array}{rcl}
Qx\,(G \wedge F) & \Rightarrow_{MS} & Qx\, G \wedge F \\
Qx\,(G \vee F) & \Rightarrow_{MS} & Qx\, G \vee F \\
\forall x\,(G \wedge H) & \Rightarrow_{MS} & \forall x\, G \wedge \forall x\, H \\
\exists x\,(G \vee H) & \Rightarrow_{MS} & \exists x\, G \vee \exists x\, H
\end{array}$$

Variable Renaming

Rename all variables in F such that there are no two different positions p, q with $F/p = Qx\, G$ and $F/q = Qx\, H$.

Standard Skolemization

Let F be the overall formula, then apply the rewrite rule:

$$\begin{array}{lll} \exists x\, H & \Rightarrow_{SK} & H[f(y_1, \ldots, y_n)/x] \\ & & \text{if } F/p = \exists x\, H \text{ and } p \text{ has minimal length,} \\ & & \{y_1, \ldots, y_n\} \text{ are the free variables in } \exists x\, H, \\ & & f \text{ is a new function symbol, } \mathsf{arity}(f) = n \end{array}$$

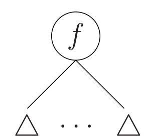
3.7 Herbrand Interpretations

From now an we shall consider PL without equality. Ω shall contains at least one constant symbol.

A *Herbrand interpretation* (over Σ) is a Σ-algebra $\mathcal{A}$ such that

- $U_{\mathcal{A}} = \mathrm{T}_\Sigma$ (= the set of ground terms over Σ)
- $f_{\mathcal{A}} : (s_1, \ldots, s_n) \mapsto f(s_1, \ldots, s_n),\ f \in \Omega, \mathsf{arity}(f) = n$

$f_{\mathcal{A}}(\triangle, \ldots, \triangle) =$



In other words, *values are fixed* to be ground terms and *functions are fixed* to be the *term constructors*. Only predicate symbols $P \in \Pi$, $\mathsf{arity}(P) = m$ may be freely interpreted as relations $P_{\mathcal{A}} \subseteq \mathrm{T}_\Sigma^m$.

Proposition 3.12 *Every set of ground atoms I uniquely determines a Herbrand interpretation $\mathcal{A}$ via*

$$(s_1, \ldots, s_n) \in P_{\mathcal{A}} \quad :\Leftrightarrow \quad P(s_1, \ldots, s_n) \in I$$