Proving Termination: Monotone Mappings

Let \((A, >_A)\) and \((B, >_B)\) be partial orderings. A mapping \(\varphi : A \rightarrow B\) is called monotone, if \(a >_A a'\) implies \(\varphi(a) >_B \varphi(a')\) for all \(a, a' \in A\).

**Lemma 4.10** If \(\varphi : A \rightarrow B\) is a monotone mapping from \((A, >_A)\) to \((B, >_B)\) and \((B, >_B)\) is well-founded, then \((A, >_A)\) is well-founded.

### 4.3 Rewrite Systems

Let \(E\) be a set of equations.

The rewrite relation \(\rightarrow_E \subseteq T_\Sigma(X) \times T_\Sigma(X)\) is defined by

\[
s \rightarrow_E t \iff \text{there exist } (l \approx r) \in E, p \in \text{pos}(s),\n\quad \text{and } \sigma : X \rightarrow T_\Sigma(X),\n\quad \text{such that } s/p = l\sigma \text{ and } t = s[r\sigma]_p.\n\]

An instance of the lhs (left-hand side) of an equation is called a redex (reducible expression). Contracting a redex means replacing it with the corresponding instance of the rhs (right-hand side) of the rule.

An equation \(l \approx r\) is also called a rewrite rule, if \(l\) is not a variable and \(\text{var}(l) \supseteq \text{var}(r)\).

Notation: \(l \rightarrow r\).

A set of rewrite rules is called a term rewrite system (TRS).

We say that a set of equations \(E\) or a TRS \(R\) is terminating, if the rewrite relation \(\rightarrow_E\) or \(\rightarrow_R\) has this property.

(Analogously for other properties of abstract reduction systems).

Note: If \(E\) is terminating, then it is a TRS.

**E-Algebras**

Let \(E\) be a set of equations. A \(\Sigma\)-algebra \(A\) is called an \(E\)-algebra, if \(A \models \forall \vec{x}(s \approx t)\) for all \(\forall \vec{x}(s \approx t) \in E\).

If \(E \models \forall \vec{x}(s \approx t)\) (i.e., \(\forall \vec{x}(s \approx t)\) is valid in all \(E\)-algebras), we write this also as \(s \approx_E t\).

Goal:
Use the rewrite relation \(\rightarrow_E\) to express the semantic consequence relation syntactically:

\[
s \approx_E t \text{ if and only if } s \rightarrow^*_E t.\n\]
Let $E$ be a set of equations over $T_{\Sigma}(X)$. The following inference system allows to derive consequences of $E$:

$\vdash t \approx t$ \hspace{1cm} (Reflexivity)

$\vdash t \approx t'$ \hspace{1cm} (Symmetry)

$\vdash t' \approx t$ \hspace{1cm} (Transitivity)

$\vdash f(t_1, \ldots, t_n) \approx f(t'_1, \ldots, t'_n)$ \hspace{1cm} (Congruence)

$\vdash t \sigma \approx t' \sigma$ \hspace{1cm} (Instance)

if $(t \approx t') \in E$ and $\sigma : X \rightarrow T_{\Sigma}(X)$

Lemma 4.11 The following properties are equivalent:

(i) $s \leftrightarrow^* E t$

(ii) $E \vdash s \approx t$ is derivable.

(Proof Sketch Follows)

Proof. (i)$\Rightarrow$(ii): $s \leftrightarrow^* E t$ implies $E \vdash s \approx t$ by induction on the depth of the position where the rewrite rule is applied; then $s \leftrightarrow^* E t$ implies $E \vdash s \approx t$ by induction on the number of rewrite steps in $s \leftrightarrow^* E t$.

(ii)$\Rightarrow$(i): By induction on the size (number of symbols) of the derivation for $E \vdash s \approx t$.

Constructing a quotient algebra:

Let $X$ be a set of variables.

For $t \in T_{\Sigma}(X)$ let $[t] = \{ t' \in T_{\Sigma}(X) \mid E \vdash t \approx t' \}$ be the congruence class of $t$.

Define a $\Sigma$-algebra $T_{\Sigma}(X)/E$ (abbreviated by $T$) as follows:

$U_T = \{ [t] \mid t \in T_{\Sigma}(X) \}$.

$f_T([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$ for $f \in \Omega$.

Lemma 4.12 $f_T$ is well-defined: If $[t_i] = [t'_i]$, then $[f(t_1, \ldots, t_n)] = [f(t'_1, \ldots, t'_n)]$.

Proof. Follows directly from the Congruence rule for $\vdash$. □
Lemma 4.13 \( T = T_\Sigma(X)/E \) is an \( E \)-algebra. (Proof Follows)

**Proof.** Let \( \forall x_1 \ldots x_n(s \approx t) \) be an equation in \( E \); let \( \beta \) be an arbitrary assignment.

We have to show that \( T(\beta)(\forall \vec{x}(s \approx t)) = 1 \), or equivalently, that \( T(\gamma)(s) = T(\gamma)(t) \) for all \( \gamma = \beta[x_i \mapsto [t_i] | 1 \leq i \leq n] \) with \([t_i] \in U_T\).

Let \( \sigma = [t_1/x_1, \ldots, t_n/x_n] \), then \( s\sigma \in T(\gamma)(s) \) and \( t\sigma \in T(\gamma)(t) \).

By the Instance rule, \( E \vdash s\sigma \approx t\sigma \) is derivable, hence \( T(\gamma)(s) = [s\sigma] = [t\sigma] = T(\gamma)(t) \). \( \square \)

Lemma 4.14 Let \( X \) be a countably infinite set of variables; let \( s, t \in T_\Sigma(X) \). If \( T_\Sigma(X)/E \models \forall \vec{x}(s \approx t) \), then \( E \vdash s \approx t \) is derivable. (Proof Follows)

**Proof.** Assume that \( T \models \forall \vec{x}(s \approx t) \), i.e., \( T(\beta)(\forall \vec{x}(s \approx t)) = 1 \). Consequently, \( T(\gamma)(s) = T(\gamma)(t) \) for all \( \gamma = \beta[x_i \mapsto [t_i] | 1 \leq i \leq n] \) with \([t_i] \in U_T\).

Choose \( t_i = x_i \), then \([s] = T(\gamma)(s) = T(\gamma)(t) = [t] \), so \( E \vdash s \approx t \) is derivable by definition of \( T \). \( \square \)

Theorem 4.15 ("Birkhoff’s Theorem") Let \( X \) be a countably infinite set of variables, let \( E \) be a set of (universally quantified) equations. Then the following properties are equivalent for all \( s, t \in T_\Sigma(X) \):

(i) \( s \leftrightarrow_E t \).

(ii) \( E \vdash s \approx t \) is derivable.

(iii) \( s \approx_E t \), i.e., \( E \models \forall \vec{x}(s \approx t) \).

(iv) \( T_\Sigma(X)/E \models \forall \vec{x}(s \approx t) \).

**Proof.** (i)\(\equiv\)(ii): Lemma 4.11.

(ii)\(\Rightarrow\)(iii): By induction on the size of the derivation for \( E \vdash s \approx t \).

(iii)\(\Rightarrow\)(iv): Obvious, since \( T = T_\Sigma(X)/E \) is an \( E \)-algebra.

(iv)\(\Rightarrow\)(ii): Lemma 4.14. \( \square \)
Universal Algebra

$T_\Sigma(X)/E = T_\Sigma(X)/\approx_E = T_\Sigma(X)/\leftrightarrow^*_E$ is called the free $E$-algebra with generating set $X/\approx_E = \{ [x] \mid x \in X \}$:

Every mapping $\varphi : X/\approx_E \to B$ for some $E$-algebra $B$ can be extended to a homomorphism $\hat{\varphi} : T_\Sigma(X)/E \to B$.

$T_\Sigma(\emptyset)/E = T_\Sigma(\emptyset)/\approx_E = T_\Sigma(\emptyset)/\leftrightarrow^*_E$ is called the initial $E$-algebra.

$\approx_E = \{ (s, t) \mid E \models s \approx t \}$ is called the equational theory of $E$.

$\approx^I_E = \{ (s, t) \mid T_\Sigma(\emptyset)/E \models s \approx t \}$ is called the inductive theory of $E$.

Example:

Let $E = \{ \forall x(x + 0 \approx x), \forall x\forall y(x + s(y) \approx s(x + y)) \}$. Then $x + y \approx^I_E y + x$, but $x + y \not\approx_E y + x$.

Rewrite Relations

**Corollary 4.16** If $E$ is convergent (i.e., terminating and confluent), then $s \approx_E t$ if and only if $s \leftrightarrow^*_E t$ if and only if $s \downarrow_E = t \downarrow_E$.

**Corollary 4.17** If $E$ is finite and convergent, then $\approx_E$ is decidable.

Reminder:
If $E$ is terminating, then it is confluent if and only if it is locally confluent.

Problems:

Show local confluence of $E$.

Show termination of $E$.

Transform $E$ into an equivalent set of equations that is locally confluent and terminating.