Proving Termination: Monotone Mappings

Let $(A, >_A)$ and $(B, >_B)$ be partial orderings. A mapping $\varphi : A \to B$ is called *monotone*, if $a >_A a'$ implies $\varphi(a) >_B \varphi(a')$ for all $a, a' \in A$.

Lemma 4.10 *If $\varphi : A \to B$ is a monotone mapping from $(A, >_A)$ to $(B, >_B)$ and $(B, >_B)$ is well-founded, then $(A, >_A)$ is well-founded.*

4.3 Rewrite Systems

Let E be a set of equations.

The *rewrite relation* $\to_E \subseteq \mathrm{T}_\Sigma(X) \times \mathrm{T}_\Sigma(X)$ is defined by

$$s \to_E t \quad \text{iff} \quad \begin{array}{l} \text{there exist } (l \approx r) \in E,\ p \in \mathrm{pos}(s), \\ \text{and } \sigma : X \to \mathrm{T}_\Sigma(X), \\ \text{such that } s/p = l\sigma \text{ and } t = s[r\sigma]_p. \end{array}$$

An instance of the lhs (left-hand side) of an equation is called a *redex* (reducible expression). *Contracting* a redex means replacing it with the corresponding instance of the rhs (right-hand side) of the rule.

An equation $l \approx r$ is also called a *rewrite rule*, if l is not a variable and $\mathrm{var}(l) \supseteq \mathrm{var}(r)$.

Notation: $l \to r$.

A set of rewrite rules is called a *term rewrite system (TRS)*.

We say that a set of equations E or a TRS R is terminating, if the rewrite relation $\to_E$ or $\to_R$ has this property.

(Analogously for other properties of abstract reduction systems).

Note: If E is terminating, then it is a TRS.

E-Algebras

Let E be a set of equations. A Σ-algebra $\mathcal{A}$ is called an *E-algebra*, if $\mathcal{A} \models \forall\vec{x}(s \approx t)$ for all $\forall\vec{x}(s \approx t) \in E$.

If $E \models \forall\vec{x}(s \approx t)$ (i. e., $\forall\vec{x}(s \approx t)$ is valid in all E-algebras), we write this also as $s \approx_E t$.

Goal:
Use the rewrite relation $\to_E$ to express the semantic consequence relation syntactically:

$s \approx_E t$ if and only if $s \leftrightarrow_E^* t$.

Let E be a set of equations over $\mathrm{T}_\Sigma(X)$. The following inference system allows to derive consequences of E:

$$E \vdash t \approx t \qquad \textit{(Reflexivity)}$$

$$\frac{E \vdash t \approx t'}{E \vdash t' \approx t} \qquad \textit{(Symmetry)}$$

$$\frac{E \vdash t \approx t' \qquad E \vdash t' \approx t''}{E \vdash t \approx t''} \qquad \textit{(Transitivity)}$$

$$\frac{E \vdash t_1 \approx t'_1 \quad \ldots \quad E \vdash t_n \approx t'_n}{E \vdash f(t_1, \ldots, t_n) \approx f(t'_1, \ldots, t'_n)} \qquad \textit{(Congruence)}$$

$$E \vdash t\sigma \approx t'\sigma \qquad \textit{(Instance)}$$
if $(t \approx t') \in E$ and $\sigma : X \to \mathrm{T}_\Sigma(X)$

Lemma 4.11 *The following properties are equivalent:*

(i) $s \leftrightarrow_E^* t$

(ii) $E \vdash s \approx t$ *is derivable.*

(Proof Scetch Follows)

Proof. (i)⇒(ii): $s \leftrightarrow_E t$ implies $E \vdash s \approx t$ by induction on the depth of the position where the rewrite rule is applied; then $s \leftrightarrow_E^* t$ implies $E \vdash s \approx t$ by induction on the number of rewrite steps in $s \leftrightarrow_E^* t$.

(ii)⇒(i): By induction on the size (number of symbols) of the derivation for $E \vdash s \approx t$. □

Constructing a *quotient algebra:*

Let X be a set of variables.

For $t \in \mathrm{T}_\Sigma(X)$ let $[t] = \{ t' \in \mathrm{T}_\Sigma(X) \mid E \vdash t \approx t' \}$ be the *congruence class* of t.

Define a Σ-algebra $\mathrm{T}_\Sigma(X)/E$ (abbreviated by $\mathcal{T}$) as follows:

$U_\mathcal{T} = \{ [t] \mid t \in \mathrm{T}_\Sigma(X) \}$.

$f_\mathcal{T}([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$ for $f \in \Omega$.

Lemma 4.12 $f_\mathcal{T}$ *is well-defined: If* $[t_i] = [t'_i]$, *then* $[f(t_1, \ldots, t_n)] = [f(t'_1, \ldots, t'_n)]$.

Proof. Follows directly from the *Congruence* rule for $\vdash$. □

Lemma 4.13 *$\mathcal{T} = \mathrm{T}_\Sigma(X)/E$ is an E-algebra. (Proof Follows)*

Proof. Let $\forall x_1 \ldots x_n(s \approx t)$ be an equation in E; let β be an arbitrary assignment.

We have to show that $\mathcal{T}(\beta)(\forall \vec{x}(s \approx t)) = 1$, or equivalently, that $\mathcal{T}(\gamma)(s) = \mathcal{T}(\gamma)(t)$ for all $\gamma = \beta[\, x_i \mapsto [t_i] \mid 1 \leq i \leq n\,]$ with $[t_i] \in U_{\mathcal{T}}$.

Let $\sigma = [t_1/x_1, \ldots, t_n/x_n]$, then $s\sigma \in \mathcal{T}(\gamma)(s)$ and $t\sigma \in \mathcal{T}(\gamma)(t)$.

By the *Instance* rule, $E \vdash s\sigma \approx t\sigma$ is derivable, hence $\mathcal{T}(\gamma)(s) = [s\sigma] = [t\sigma] = \mathcal{T}(\gamma)(t)$. □

Lemma 4.14 *Let X be a countably infinite set of variables; let $s, t \in \mathrm{T}_\Sigma(X)$. If $\mathrm{T}_\Sigma(X)/E \models \forall \vec{x}(s \approx t)$, then $E \vdash s \approx t$ is derivable. (Proof Follows)*

Proof. Assume that $\mathcal{T} \models \forall \vec{x}(s \approx t)$, i.e., $\mathcal{T}(\beta)(\forall \vec{x}(s \approx t)) = 1$. Consequently, $\mathcal{T}(\gamma)(s) = \mathcal{T}(\gamma)(t)$ for all $\gamma = \beta[\, x_i \mapsto [t_i] \mid 1 \leq i \leq n\,]$ with $[t_i] \in U_{\mathcal{T}}$.

Choose $t_i = x_i$, then $[s] = \mathcal{T}(\gamma)(s) = \mathcal{T}(\gamma)(t) = [t]$, so $E \vdash s \approx t$ is derivable by definition of $\mathcal{T}$. □

Theorem 4.15 ("Birkhoff's Theorem") *Let X be a countably infinite set of variables, let E be a set of (universally quantified) equations. Then the following properties are equivalent for all $s, t \in \mathrm{T}_\Sigma(X)$:*

(i) $s \leftrightarrow_E^* t$.

(ii) $E \vdash s \approx t$ *is derivable.*

(iii) $s \approx_E t$, *i.e.,* $E \models \forall \vec{x}(s \approx t)$.

(iv) $\mathrm{T}_\Sigma(X)/E \models \forall \vec{x}(s \approx t)$.

Proof. (i)⇔(ii): Lemma 4.11.

(ii)⇒(iii): By induction on the size of the derivation for $E \vdash s \approx t$.

(iii)⇒(iv): Obvious, since $\mathcal{T} = \mathrm{T}_\Sigma(X)/E$ is an E-algebra.

(iv)⇒(ii): Lemma 4.14. □

Universal Algebra

$T_\Sigma(X)/E = T_\Sigma(X)/{\approx_E} = T_\Sigma(X)/{\leftrightarrow_E^*}$ is called the *free E-algebra* with generating set $X/{\approx_E} = \{\, [x] \mid x \in X \,\}$:

Every mapping $\varphi : X/{\approx_E} \to \mathcal{B}$ for some E-algebra $\mathcal{B}$ can be extended to a homomorphism $\hat{\varphi} : T_\Sigma(X)/E \to \mathcal{B}$.

$T_\Sigma(\emptyset)/E = T_\Sigma(\emptyset)/{\approx_E} = T_\Sigma(\emptyset)/{\leftrightarrow_E^*}$ is called the *initial E-algebra*.

${\approx_E} = \{\, (s,t) \mid E \models s \approx t \,\}$ is called the *equational theory* of E.

${\approx_E^I} = \{\, (s,t) \mid T_\Sigma(\emptyset)/E \models s \approx t \,\}$ is called the *inductive theory* of E.

Example:

Let $E = \{\forall x(x + 0 \approx x),\ \forall x \forall y(x + s(y) \approx s(x + y))\}$. Then $x + y \approx_E^I y + x$, but $x + y \not\approx_E y + x$.

Rewrite Relations

Corollary 4.16 *If E is convergent (i. e., terminating and confluent), then $s \approx_E t$ if and only if $s \leftrightarrow_E^* t$ if and only if $s{\downarrow_E} = t{\downarrow_E}$.*

Corollary 4.17 *If E is finite and convergent, then $\approx_E$ is decidable.*

Reminder:
If E is terminating, then it is confluent if and only if it is locally confluent.

Problems:

Show local confluence of E.

Show termination of E.

Transform E into an equivalent set of equations that is locally confluent and terminating.